

# COMPUTATIONS WITH THE 2D COULOMB GAS

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Bachelor's thesis  
2019:K7



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## Abstract

A somewhat self-contained exposition of the 2D Coulomb gas is presented, complete from the Gibbs measure and orthogonal polynomials to expressions for the one-point intensity and partition functions. For the model case of the Mittag-Leffler ensemble, some previously known results on the correlation kernel are reproduced, while for the less well-known lemniscate ensemble, novel numerical investigations into the one-point intensity function near 0 are made. Lastly, the logarithm of the partition function of the Mittag-Leffler ensemble is calculated up to  $O(\log n)$ , validating earlier more general results by Zabrodin and Wiegmann. A correction to the partition function of the corresponding hard edge ensemble is also derived with the novel result of a previously unseen  $\sqrt{n}$  term in the expansion of  $\log Z_n$ .

*Throughout this work it has been my firm intention to give reference to the stated results and credit to the work of others. All theorems, propositions, lemmas and examples left unmarked are assumed to be too well-known for a reference to be given.*



## Populärvetenskaplig sammanfattning

Den tvådimensionella Coulombgasen består av en samling repellerande laddade partiklar som lever i planet och påverkas av ett instängande fält som hindrar gasen från att upplösas. Om vi ökar antalet partiklar så kommer de att bli mer och mer obekväma med att vara nära varandra och det krävs då ett starkare fält för att förhindra att gasen skingras. Det visar sig att för en ganska bred samling instängningar så kommer gasen att stabiliseras när antalet partiklar går mot oändligheten i meningen att nästan alla partiklarna kommer befinna sig i ett visst begränsat område som vi kan tänka oss som en tvådimensionell bubbla. Ett nödvändigt villkor för det är att instängningen skalas tillsammans med antalet partiklar på så sätt att dubbelt så många partiklar innebär dubbelt så stark instängning.

Allmänt så är stora partikelsystem såsom detta svåra att lösa exakt så att man vet var varje partikel befinner sig. Istället försöker vi beskriva mer statistiska kvantiteter såsom hur stor andel av gasen vi förväntar oss hitta utanför bubblan eller vad den förväntade densiteten hos partiklarna är i en given punkt. För att göra alla kvantiteter mer exakta och för att undvika att göra några arbiträra val så låter vi antalet partiklar gå mot oändligheten och då har dessa frågor exakta svar.

Medan denna konstruktion kan tyckas vara arbiträr så visar det sig att sättet som vi beskriver systemet statistiskt inte är unikt för Coulombgasen. Om vi exempelvis tar ett komplext polynom, en funktion av formen  $f(z) = z^n + a_{n-1}z^{n-1} + \dots + a_1z + a_0$ , och låter koefficienterna  $a_0, \dots, a_{n-1}$  vara normalfördelade så kommer nollställena till  $f$  repellera varandra på ett sätt som liknar gasen. Detsamma är även sant för egenvärdena hos vissa typer av slumpmatriser.

I detta arbete går vi igenom mycket klassisk teori kring hur man matematiskt kan hantera dessa typer av system, samt undersöker en del system mer exakt i hopp om att kunna komma fram till resultat som kan generaliseras. Med hjälp av numeriska beräkningar bestämmer vi gasens täthet i ett område kring en spetsig kant, något som ingen har lyckats lösa exakt matematiskt ännu. Vi undersöker även hur en viss central statistisk kvantitet ändras när vi inte tillåter partiklarna att lämna den tvådimensionella bubblan.



## Acknowledgements

I would like to wholeheartedly thank my supervisor Yacin Ameer for his dedication and support throughout the project as well as his willingness to share ideas which has been very appreciated. I am also very thankful to Amanda Ledell for making my first couple of years in Lund much more enjoyable.

Simon Halvdansson





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# Introduction

There are really two main ways of viewing the 2D Coulomb gas; to the physicist it is the statistical ensemble of a system of electrons in the plane subject to some confining potential  $Q$  which prohibits dispersion while to the statistician, it is a point process on the complex plane picked with respect to a rather ugly probability measure (depending on a function  $Q$ ) which comes up in applications surprisingly often. It is not self-evident from the setup but as the number of points or particles goes to infinity, the system exhibits emergent phenomena in that on average the points tend to be distributed in accordance with some not-too-complicated function. The question is now how we can find and study such a function.

From the more physical side, the language of potential theory is well suited for this problem once we consider the charges to be smeared out and the distribution given by some density function instead of a finite set of points. A classical theorem then gives a result with great resemblance to Gauss' law in differential form,  $\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$ , which gives us a lot of information on the limiting macroscopic distribution of the system. Potential theory does however only tell us things about the limiting behavior and nothing about the behavior as the number of particles increases. For this, point processes are more appropriate and since they have been studied for a long time in probability theory, there are powerful tools available already. Arguably the most powerful such tools is the *one-point intensity function* which for any number of points gives the expected density at some location. As one might guess, determining the one-point intensity function is highly nontrivial, especially for arbitrary potentials  $Q$ .

The goal of this thesis is two-fold; the basic theory leading up to the one-point intensity function is not very well documented in the literature, something which this thesis aims to address. Secondly, open and new problems are discussed in Chapters 3 and 4, providing some new insights and conjectures. Overall the thesis aims to provide a satisfactory introduction to the 2D Coulomb gas with computations of several important properties of the system. The reader is expected to have a good grasp of basic concepts from introductory real analysis, complex analysis and linear algebra as well as knowledge about some basic notions from measure theory, probability theory and Fourier analysis.



# Chapter 1

## Theoretical background

### 1.1 Setup and outline

Fix a *potential*  $Q : \mathbb{C} \rightarrow \mathbb{R} \cup \{+\infty\}$  and consider a random configuration (*system, ensemble, point process*)  $\{\zeta_j\}_{j=1}^n$  picked with respect to the probability measure  $\mathbb{P}_n$  on  $\mathbb{C}^n$  which is given by

$$d\mathbb{P}_n = \frac{1}{Z_n} e^{-H_n} dV_n, \quad H_n = \sum_{i \neq j} \log \frac{1}{|\zeta_i - \zeta_j|} + n \sum_{j=1}^n Q(\zeta_j) \quad (1.1)$$

where  $Z_n$ , the *partition function*, is a constant ensuring  $\mathbb{P}_n(\mathbb{C}^n) = 1$  and  $dV_n$  is the Lebesgue measure on  $\mathbb{C}^n$  divided by  $\pi^n$ . Under suitable conditions on  $Q$  (loosely  $Q(\zeta) \rightarrow \infty$  as  $\zeta \rightarrow \infty$ ), the system will stabilize in the  $n \rightarrow \infty$  limit in that the points will tend to occupy only a certain compact set  $S \subset \mathbb{C}$  known as the *droplet*. An example of this is shown in the two figures below.

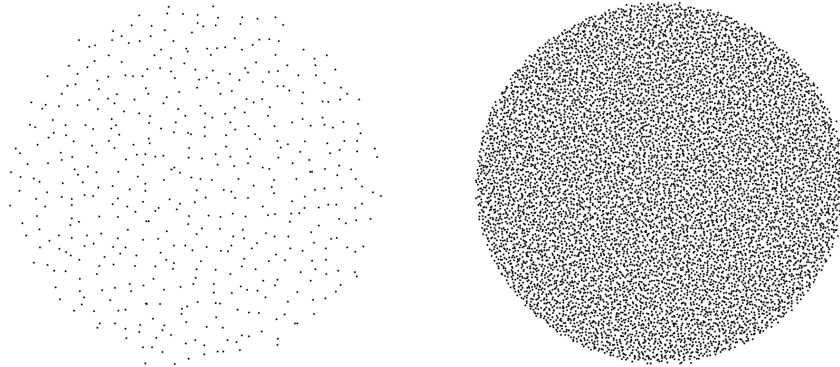


Figure 1.1: Outcomes of the point process with  $Q(\zeta) = |\zeta|^2$  and  $n = 500, 10\,000$ .

In the above example the droplet is the closed unit disk. When asking questions about the behavior of the particle distribution, the probability density function of equation (1.1) is not of much help as it gives the probability for entire systems to occupy certain states instead of the particle density. To remedy this we can introduce the *one-point intensity function* defined as

$$\mathbf{R}_n(\zeta) = \lim_{\varepsilon \rightarrow 0} \frac{n}{\varepsilon^2} \mathbb{P}_n(\mathbb{D}(\zeta, \varepsilon) \times \mathbb{C}^{n-1}) = \lim_{\varepsilon \rightarrow 0} \frac{\mathbb{E}(\# \text{ points in disk centered at } \zeta \text{ of area } \varepsilon^2)}{\varepsilon^2}$$

where  $\mathbb{E}$  is the expectation value with respect to  $\mathbb{P}_n$ . We mention that the factor  $n$  compensates for the arbitrariness of placing the disk in the first coordinate. It turns out that the one-point intensity function can be easily obtained from a certain object known as the *correlation kernel* which is related to a linear space of weighted polynomials in  $\zeta$  with the weight function depending on the potential  $Q$ .

It has been shown that  $n\Delta Q$  is a first order estimate of  $\mathbf{R}_n$  in the interior of the droplet [3, 25] and similar universality results have recently been obtained for points where  $\Delta Q > 0$  and  $\partial S$  is smooth in [26]. In this thesis we therefore probe a point where  $\Delta Q = 0$  numerically for which there is no existing framework for determining  $\mathbf{R}_n(\zeta)$ .

## Outline

In Chapter 1 we develop the required theoretical machinery to probe both local and global properties of the system. This is in the form of the partition function in Section 1.2 and the correlation kernel and the associated one-point intensity function in Section 1.3. In Section 1.4 we review some classical results describing the macroscopic distribution of the particles. Chapter 2 discusses a family of well understood ensembles and some previous results are reviewed. In Chapter 3 a less known family of ensembles are discussed and some novel numerical results are presented. The partition functions of the ensembles discussed in Chapter 2 are brought up in Chapter 4 where novel results regarding the expansion of the logarithm of the partition function  $Z_n$  are presented. Chapters 2, 3 and 4 can for the most part be read in any order.

## Notational conventions

Throughout the thesis we identify  $\mathbb{R}^2$  with  $\mathbb{C}$  for convenience. The characteristic function of a set  $E$  is denoted by  $\mathbf{1}_E$  and we work with  $1/\pi$  of the normal area element so that  $dA = \frac{1}{\pi} dx dy$ . We let  $\Delta$  denote  $1/4$  the usual Laplacian so that  $\Delta = \partial_z \partial_{\bar{z}}$  with  $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ ,  $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$ . The open disk of radius  $r$  centered at  $\zeta$  is denoted by  $\mathbb{D}(\zeta, r)$ .

## 1.2 Problem interpretation and background

The goal of this section is to shed some light on how equation (1.1) can be interpreted and why it is of interest. We also show how we can rewrite (1.1) as to make it easier to work with in the upcoming sections. The equation is repeated here for reference.

$$d\mathbb{P}_n = \frac{1}{Z_n} e^{-H_n} dV_n, \quad H_n = \sum_{i \neq j} \log \frac{1}{|\zeta_i - \zeta_j|} + n \sum_{j=1}^n Q(\zeta_j).$$

Before continuing further we present some basic preliminaries on statistical mechanics.

### 1.2.1 Statistical Mechanics

Statistical mechanics is a vast field and we just outline some important ideas used in this thesis. For a more complete overview, the reader is referred to any undergraduate textbook on statistical mechanics such as [30] or [19]. In statistical mechanics we drop the idea of solving equations of motions or considering perturbations to exact relations and instead consider statistical properties of a system. This because we are not really

interested in which particles display a certain behavior if we can't tell the particles apart in the first place.

One of the most central objects in statistical mechanics is the *Gibbs measure* which is a probability measure for the entire system. It depends only on the inverse temperature  $\beta = \frac{1}{k_B T}$  and the Hamiltonian  $H$ , where  $k_B \approx 1.4 \cdot 10^{-23}$  J/K is the *Boltzmann constant* and  $T$  is the temperature. Letting  $\Omega$  denote the space of all possible configurations for the system and  $x \in \Omega$  a specific configuration we have that the associated probability density function  $f$  is of the form

$$f(x) = \frac{1}{Z} e^{-\beta H(x)}, \quad Z = \int_{\Omega} e^{-\beta H(x)} dx \quad (1.2)$$

where  $Z$  is known as the *partition function*. In this thesis we only discuss the  $\beta = 1$  case for which significantly more powerful mathematical tools are available. We can extract a lot of useful information from the partition function, especially when we know how it depends on different parameters such as the temperature or the number of particles.

The final concept we discuss is *entropy*. To do so, we must first introduce the concept of *microstates* which is a specific configuration of a system. Multiple microstates can correspond to the same *macrostate* meaning they have the same energy, volume, temperature, etc., but differ only microscopically such as in two particles switching places. For a system where the probability of finding the system in the  $i$ :th microstates is  $p_i$ , the Gibbs entropy is defined as

$$k_B \sum_i p_i \log \frac{1}{p_i}. \quad (1.3)$$

A fundamental postulate of statistical mechanics states that at thermal equilibrium, meaning that the system has stabilized and there is no heat flow, all microstates are equally likely. It is then easy to see that the entropy takes the form  $S = k_B \log \Omega$  where  $\Omega$  is the number of microstates corresponding to the current macrostate.

### 1.2.2 The 2D Coulomb gas

We begin by noting that there is a striking similarity between (1.1) and (1.2). Indeed, if we set  $\beta = 1$  and  $H = H_n$  they are identical. From here we try to deduce properties of a system with Hamiltonian  $H_n$ . Such a system must be in equilibrium as there is no kinetic energy term. We claim that the two sums can be seen as representing the pairwise Coulomb repulsion and total potential energy of a collection of pointlike charges in  $\mathbb{C}$  located at  $\{\zeta_j\}_{j=1}^n$  subject to a potential  $Q$ . Indeed, scaling the potential with  $n$  is no issue and as for  $-\log |\zeta_i - \zeta_j|$  representing the energy associated with two charged particles at  $\zeta_i$  and  $\zeta_j$ , the next proposition shows that this is the solution to Poisson's equation modulo a factor  $\frac{1}{2}$ . We remark that this is a well-known proposition which is included for completion. An alternative proof can be found in e.g. [14].

**Proposition 1.2.1.**  $\varphi(z) = \frac{1}{2} \log |z|$  solves Poisson's equation  $\Delta \varphi = \delta$  on  $\mathbb{C}$  in the distributional sense.

*Proof.* Proving the assertion in the distributional sense means that we need to prove that

$$\int_{\mathbb{C}} \Delta \log |z| \cdot \phi(z) dA(z) = 2\phi(0)$$

for all functions  $\phi \in C^\infty$  with compact support. Let  $D$  be a domain such that  $\text{supp } \phi \subset D$ . Then clearly

$$\int_{\mathbb{C}} \Delta \log |z| \cdot \phi(z) dA(z) = \int_D \Delta \log |z| \cdot \phi(z) dA(z)$$

since  $\phi$  vanishes outside  $D$ . Recall Green's identity which states that

$$\int_E (u\Delta v - v\Delta u) dA = \frac{1}{\pi} \int_{\partial E} \left( u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) ds.$$

Setting  $E = D$ ,  $v(z) = \log |z|$  and  $u = \phi$  we have that

$$\int_D \Delta \log |z| \cdot \phi(z) dA(z) = \int_D \log |z| \cdot \Delta \phi(z) dA(z)$$

since  $\phi$  vanishes near  $\partial D$  also. Let  $D_\varepsilon = D \setminus \mathbb{D}(0, \varepsilon)$  be  $D$  with a disk of radius  $\varepsilon$  stamped out around 0. Then

$$\int_D \log |z| \cdot \Delta \phi(z) dA(z) = \lim_{\varepsilon \rightarrow 0} \int_{D_\varepsilon} \log |z| \cdot \Delta \phi(z) dA(z). \quad (1.4)$$

We now turn to computing the right hand side of the above expression. Invoking Green's identity again with  $u(z) = \log |z|$  and  $v = \phi$  we find

$$\begin{aligned} \int_{D_\varepsilon} \log |z| \cdot \Delta \phi(z) dA(z) &= \int_{D_\varepsilon} \Delta \log |z| \cdot \phi(z) dA(z) \\ &+ \frac{1}{\pi} \int_{\partial D_\varepsilon} \left( \log |z| \frac{\partial \phi(z)}{\partial n} - \phi(z) \frac{\partial \log |z|}{\partial n} \right) ds. \end{aligned}$$

We have that  $\log |z|$  is harmonic in  $D_\varepsilon$  by

$$\Delta \log |z| = \Delta \frac{1}{2} \log |z|^2 = \partial_z \partial_{\bar{z}} \frac{1}{2} (\log z + \log \bar{z}) = 0$$

and so the first integral vanishes. When integrating along  $\partial D_\varepsilon$ , the part along  $\partial D$  will disappear since  $\phi$  vanishes there. The first term will disappear in the  $\varepsilon \rightarrow 0$  limit as  $|z| = \varepsilon$ ,  $\left| \frac{\partial \phi}{\partial n} \right| \leq C$  for some  $C \in \mathbb{R}$  since  $\phi \in C^\infty$  and  $\varepsilon \log \varepsilon \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . The normal in the second term is directed inwards and so we have

$$\int_{D_\varepsilon} \log |z| \cdot \Delta \phi(z) dA(z) = \frac{1}{\pi} \frac{1}{\varepsilon} \int_{|z|=\varepsilon} \phi(z) ds \rightarrow \frac{2\pi\varepsilon}{\pi\varepsilon} \phi(0) = 2\phi(0)$$

as  $\varepsilon \rightarrow 0$  as  $\phi$  is continuous. In view of (1.4), the result follows.  $\square$

*Remark.* We say that  $\frac{1}{2} \log |z|$  is the *fundamental solution of the Laplace operator in  $\mathbb{C}$* .

This type of system is known as a *2D Coulomb gas*, *log gas*, *one component plasma* or  *$\beta$ -ensemble* among other names and the scaled potential  $nQ$  is referred to as the *external field*. Since this type of system can only exist in two dimensions, we can think of it as being in the cross section of an infinitely long conductor in which forces along the direction of the conductor cancel out.

An important area where the 2D Coulomb gas is used is Ginzburg-Landau Vortices which is a type of phenomenon in a model of superconductivity known as Ginzburg-Landau theory. See e.g. [35] for more on this. For other applications, [1] is an excellent source.



## Simulating the gas

Taking the 2D Coulomb gas analogue one step further, we can discard the statistical mechanics interpretation and simulate a system of  $n$  particles with Hamiltonian  $H_n$ . This could be seen as setting  $\beta = \infty$  although in practice we won't be able to verify that our simulation has reached a *global* minimum when it stabilizes. For high enough  $n$ , we should expect these locally optimal configurations we show high resemblance to an actual observation of  $\{\zeta_j\}_{j=1}^n$ . The four configurations below are all at  $n = 5000$  and the same scale.

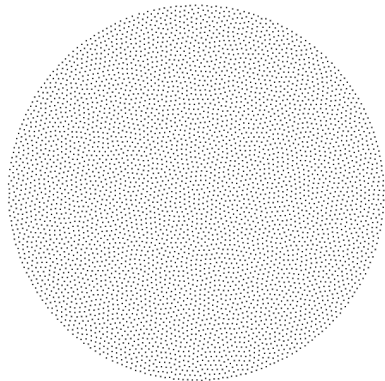


Figure 1.2:  $Q(\zeta) = |\zeta|^2$ .

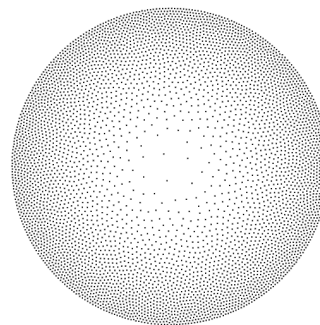


Figure 1.3:  $Q(\zeta) = |\zeta|^4$ .

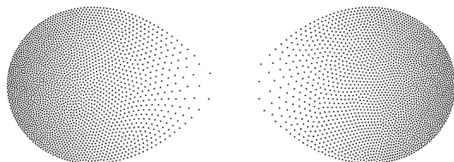


Figure 1.4:  $Q(\zeta) = |\zeta|^4 - \frac{2}{\sqrt{2}} \operatorname{Re}(\zeta^2)$ .

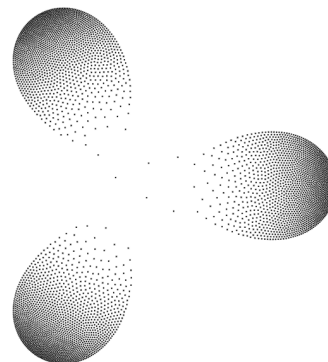


Figure 1.5:  $Q(\zeta) = |\zeta|^6 - \frac{2}{\sqrt{5}} \operatorname{Re}(\zeta^3)$ .

The potentials in Figures 1.2 and 1.3 will be investigated further in Chapters 2 and 4 and those in Figures 1.4 and 1.5 in Chapter 3.

The code for generating the above figures can be found in [23].

*Remark.* Global minimizers of the energy are known as *weighted Fekete configurations* in the literature.

### 1.2.3 Random matrices

In random matrix theory, properties of  $n \times n$  matrices such as their spectra are investigated. The study of random matrices began in the 1950's when Wigner attempted

to approximate the Hamiltonian of a heavy nucleus using a random matrix [1]. The 2D Coulomb gas appears as a natural analogy for random matrices due to the fact that close relatives to (1.1) appear when endowing a random matrices with a probability measure. This connection was used early in the development of random matrices, such as in [16].

Recall that a matrix  $M$  is said to be *normal* if  $MM^* = M^*M$  where  $M^*$  denotes the conjugate transpose of  $M$ . Without going into too much detail we state that in *normal matrix models*, the probability distribution is of the form

$$\frac{1}{Z_n} e^{-n \text{Tr}(Q(M))} dM, \quad Z_n = \int e^{-n \text{Tr}(Q(M))} dM$$

where  $dM$  is a sort of “volume form” on the space of normal  $n \times n$  matrices and  $\text{Tr}$  is the trace. We need for the matrix to be normal in order for the trace of a function of the matrix to be well behaved. This can be seen by noting that for any polynomial  $p$  we have

$$p(M) = \sum_{k=0}^n a_k (UDU^*)^k = \sum_{k=0}^n a_k UD^k U^* = U \left( \sum_{k=0}^n a_k D^k \right) U^* = U p(D) U^*$$

where we used that normal matrices can be diagonalized as  $M = UDU^*$  where  $U$  is unitary and  $D = \text{diag}(\lambda_k)$  where  $\lambda_1, \dots, \lambda_n$  are the eigenvalues of  $M$ . In particular, if  $M$  is a  $n \times n$  normal matrix and  $Q$  is any function, we can find a polynomial  $p$  of degree  $n$  such that  $p(\lambda_k) = Q(\lambda_k)$  for  $k = 1, \dots, n$ . Combining this with the fact that  $\text{Tr}(AB) = \text{Tr}(BA)$  we find

$$\text{Tr}(Q(M)) = \text{Tr}(Q(UDU^*)) = \text{Tr}(Q(D)) = \sum_{k=1}^n Q(\lambda_k).$$

The change of variables to the eigenvalues introduces a Jacobian corresponding to the factor  $\prod_{i \neq j} |\lambda_i - \lambda_j|$  in (1.1) [28]. See e.g [12, 17] or [8] and references therein, in particular [32], for more details on the random normal matrix interpretation.

We finally mention that for the model case of the Ginibre ensemble for which  $Q(\zeta) = |\zeta|^2$ , the 2D Coulomb gas with  $n$  particles corresponds exactly to the case of a  $n \times n$  matrix where each element is identically and independently distributed according to a complex Gaussian with variance  $1/n$  [21]. Matrices where the elements are identical independent random variables are known as *Wigner matrices* and are of course not normal in general but it still turns out that we get the eigenvalue distribution of an appropriate normal matrix in the special Ginibre case. This fact was used to generate Figure 1.1. For more in general about the connection between random matrices and the 2D Coulomb gas and classical proofs, see [20].

We emphasize that it is only the  $\beta = 1$  case which corresponds to the random normal matrix model.

#### 1.2.4 Vandermonde determinant and the partition function

Our goal of this section is to rewrite (1.1) to make it easier to work with. We begin by noting that

$$\sum_{i \neq j} \log \frac{1}{|\zeta_i - \zeta_j|} = 2 \sum_{1 \leq i < j \leq n} \log \frac{1}{|\zeta_i - \zeta_j|}.$$

Substituting this into (1.1) we have

$$\begin{aligned} d\mathbb{P}_n &= \frac{1}{Z_n} e^{-H_n(\zeta_1, \dots, \zeta_n)} dV_n = \frac{1}{Z_n} \prod_{1 \leq i < j \leq n} |\zeta_j - \zeta_i|^2 e^{-n(Q(\zeta_1) + \dots + Q(\zeta_n))} dV_n \\ &= \frac{1}{Z_n} \prod_{1 \leq i < j \leq n} |\zeta_j - \zeta_i|^2 d\mu_n(\zeta_1) \cdots d\mu_n(\zeta_n), \quad d\mu_n(\zeta_k) = e^{-nQ(\zeta_k)} dA(\zeta_k). \end{aligned} \quad (1.5)$$

To rewrite the above product we make use of the *Vandermonde determinant* which is a well-known tool from linear algebra with the following special property.

**Lemma 1.2.2.** *Given a collection of complex numbers  $\zeta_1, \zeta_2, \dots, \zeta_n$  we can write*

$$\prod_{1 \leq i < j \leq n} (\zeta_j - \zeta_i) = \det V_n, \quad V_n = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \zeta_1 & \zeta_2 & \cdots & \zeta_n \\ \zeta_1^2 & \zeta_2^2 & \cdots & \zeta_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \zeta_1^{n-1} & \zeta_2^{n-1} & \cdots & \zeta_n^{n-1} \end{bmatrix} \quad (1.6)$$

where  $V_n$  is known as a *Vandermonde matrix* of order  $n$ .

*Proof.* We want the top row and the leftmost column to be all zeros except for the top left corner where we want to keep the 1. This in order to obtain a  $(n-1) \times (n-1)$  matrix of a similar form on which we can perform induction. Subtracting the leftmost column from all other columns to remove the 1's we obtain

$$\det V_n = \begin{vmatrix} 1 & 0 & \cdots & 0 \\ \zeta_1 & \zeta_2 - \zeta_1 & \cdots & \zeta_n - \zeta_1 \\ \zeta_1^2 & \zeta_2^2 - \zeta_1^2 & \cdots & \zeta_n^2 - \zeta_1^2 \\ \vdots & \vdots & \ddots & \vdots \\ \zeta_1^{n-1} & \zeta_2^{n-1} - \zeta_1^{n-1} & \cdots & \zeta_n^{n-1} - \zeta_1^{n-1} \end{vmatrix}.$$

To clean up the leftmost column, for each row starting at the bottom, we subtract  $\zeta_1$  times the above row. We are then left with the determinant

$$\begin{aligned} \det V_n &= \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & \zeta_2 - \zeta_1 & \cdots & \zeta_n - \zeta_1 \\ 0 & \zeta_2(\zeta_2 - \zeta_1) & \cdots & \zeta_n(\zeta_n - \zeta_1) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \zeta_2^{n-2}(\zeta_2 - \zeta_1) & \cdots & \zeta_n^{n-2}(\zeta_n - \zeta_1) \end{vmatrix} = \begin{vmatrix} \zeta_2 - \zeta_1 & \cdots & \zeta_n - \zeta_1 \\ \zeta_2(\zeta_2 - \zeta_1) & \cdots & \zeta_n(\zeta_n - \zeta_1) \\ \vdots & \ddots & \vdots \\ \zeta_2^{n-2}(\zeta_2 - \zeta_1) & \cdots & \zeta_n^{n-2}(\zeta_n - \zeta_1) \end{vmatrix} \\ &= \prod_{k=2}^n (\zeta_k - \zeta_1) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \zeta_2 & \zeta_3 & \cdots & \zeta_n \\ \vdots & \vdots & \ddots & \vdots \\ \zeta_2^{n-2} & \zeta_3^{n-2} & \cdots & \zeta_n^{n-2} \end{vmatrix}. \end{aligned}$$

Repeating the procedure for the new determinant we obtain the same result but with  $\zeta_1$  and  $k = 2$  replaced by  $\zeta_2$  and  $k = 3$  respectively. Continuing in this fashion we eventually obtain

$$\prod_{k=2}^n (\zeta_k - \zeta_1) \cdot \prod_{k=3}^n (\zeta_k - \zeta_2) \cdots (\zeta_n - \zeta_{n-1}) = \prod_{1 \leq i < j \leq n} (\zeta_j - \zeta_i)$$

as desired.  $\square$

It is now clear that we can write the product as

$$\prod_{1 \leq i < j \leq n} |\zeta_j - \zeta_i|^2 = |\det V_n|^2.$$

Our next step is to further rewrite the Vandermonde determinant. Consider the inner product

$$\langle p_1, p_2 \rangle = \int_{\mathbb{C}} p_1(\zeta) \overline{p_2(\zeta)} d\mu_n(\zeta) = \int_{\mathbb{C}} p_1(\zeta) \overline{p_2(\zeta)} e^{-nQ(\zeta)} dA(\zeta), \quad \|p\|^2 = \langle p, p \rangle \quad (1.7)$$

on the space of polynomials of degree strictly less than  $n$ . Now let  $(\pi_k)_{k=0}^{n-1}$  be monic orthogonal polynomials with  $\deg \pi_k = k$ . We will discuss these polynomials further in Section 1.3.2 and for now present a lemma to rewrite the Vandermonde determinant using them.

**Lemma 1.2.3.** *Let  $(\pi_k)_{k=0}^{n-1}$  be monic orthogonal polynomials with  $\deg \pi_k = k$ , then the Vandermonde determinant can be written as*

$$\det V_n = \begin{vmatrix} \pi_0(\zeta_1) & \pi_0(\zeta_2) & \cdots & \pi_0(\zeta_n) \\ \pi_1(\zeta_1) & \pi_1(\zeta_2) & \cdots & \pi_1(\zeta_n) \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{n-1}(\zeta_1) & \pi_{n-1}(\zeta_2) & \cdots & \pi_{n-1}(\zeta_n) \end{vmatrix}. \quad (1.8)$$

*Proof.* We start of from the representation in (1.6) and consider the  $k$ :th row,  $0 < k \leq n-1$ , and let  $c_i$  be the  $i$ :th coefficient of  $\pi_k$ . We want the element in the  $j$ :th column to be of the form  $c_0 + c_1\zeta_j + c_2\zeta_j^2 + \dots + c_{k-1}\zeta_j^{k-1} + \zeta_j^k$ . Add first the  $k-1$ :th row scaled as to give  $\zeta_j^{k-1}$  the correct factor  $c_{k-1}$  so that the element becomes of the form  $\zeta_j^k + c_{k-1}\zeta_j^{k-1} + O(\zeta_j^{k-2})$ . Add then the  $k-2$ :th row scaled as to give  $\zeta_j^{k-2}$  the correct factor  $c_{k-2}$ . Continue in this manner until the row is of the desired format. This can be done for all rows except for the first one but that is already a monic polynomial of degree 0 so we are done.  $\square$

*Remark.* The above proof works for any set of monic polynomials, not just  $(\pi_k)_{k=0}^{n-1}$ . This setup will however prove useful in the next theorem as well as later on in Lemma 1.3.5.

We now develop an expression for the partition function  $Z_n$ . Since the partition function guarantees total probability 1, by (1.5) it must be of the form

$$Z_n = \int_{\mathbb{C}^n} |\det V_n|^2 d\mu_n(\zeta_1) \cdots d\mu_n(\zeta_n). \quad (1.9)$$

The proof of the following theorem is based on that in [32, p. 78].

**Theorem 1.2.4.** *Let  $Z_n$  denote the partition function and  $(\pi_k)_{k=0}^{n-1}$  the monic orthogonal polynomials. Then*

$$Z_n = \|\pi_0\|^2 \cdot \|\pi_1\|^2 \cdots \|\pi_{n-1}\|^2 \cdot n! \quad (1.10)$$

with norm  $\|\cdot\|$  as in (1.7).

*Proof.* We make use of the representation of the Vandermonde determinant from Lemma 1.2.3. From the multilinearity of the determinant it follows that

$$\det V_n = \det [\pi_{i-1}(\zeta_j)]_{i,j=1}^n = \prod_{i=0}^{n-1} (\|\pi_i\|^2) \cdot \det \left[ \frac{1}{\|\pi_{i-1}\|^2} \pi_{i-1}(\zeta_j) \right]_{i,j=1}^n. \quad (1.11)$$

We use the Leibniz formula for determinants, see any book on Linear Algebra such as [27, p. 334] for a proof. Let  $i \in S_n$  denote a permutation such that  $i : k \mapsto i_k$  and  $\sigma(i)$  the sign function of permutations taking the values  $+1$  and  $-1$  when the permutation  $i$  is even and odd respectively. Then

$$\begin{aligned} |\det V_n|^2 &= \sum_{i \in S_n} \sum_{j \in S_n} \sigma(i)\sigma(j) \cdot \pi_{i_1}(\zeta_1) \cdots \pi_{i_n}(\zeta_n) \overline{\pi_{j_1}(\zeta_1)} \cdots \overline{\pi_{j_n}(\zeta_n)} \\ &= \prod_{i=0}^{n-1} (\|\pi_i\|^2) \cdot \sum_{i \in S_n} \sum_{j \in S_n} \sigma(i)\sigma(j) \prod_{k=1}^n \frac{1}{\|\pi_{i_k}\|^2} \pi_{i_k}(\zeta_k) \overline{\pi_{j_k}(\zeta_k)} \end{aligned}$$

where the last equality is justified by (1.11). We now integrate this

$$\begin{aligned} Z_n &= \int_{\mathbb{C}^n} |\det V_n|^2 d\mu_n(\zeta_1) \cdots d\mu_n(\zeta_n) \\ &= \prod_{i=0}^{n-1} (\|\pi_i\|^2) \cdot \sum_{i \in S_n} \sum_{j \in S_n} \sigma(i)\sigma(j) \prod_{k=1}^n \underbrace{\int_{\mathbb{C}} \frac{1}{\|\pi_{i_k}\|^2} \pi_{i_k}(\zeta_k) \overline{\pi_{j_k}(\zeta_k)} d\mu_n(\zeta_k)}_{=\delta_{i_k, j_k}} \end{aligned}$$

where the equality in the underbrace follows from the orthogonality of  $(\pi_k)_{k=0}^{n-1}$ . Thus only the pairs of permutations where  $i_k = j_k$  for all  $k$  will survive in the double sum. These pairs of permutations must be identical and so they have the same parity i.e.  $\sigma(i)\sigma(j) = 1$ . We run through these permutations as  $\sum_{k \in S_n}$  and find

$$Z_n = \prod_{i=0}^{n-1} \|\pi_i\|^2 \cdot \sum_{k \in S_n} \prod_{j=1}^n \delta_{k_j, k_j} = \prod_{i=0}^{n-1} \|\pi_i\|^2 \cdot \sum_{k \in S_n} 1 = \|\pi_0\|^2 \cdot \|\pi_1\|^2 \cdots \|\pi_{n-1}\|^2 \cdot n!$$

which is what we wished to show.  $\square$

We will return to the partition function in Chapter 4 and use it to determine some properties of a certain family of systems for which the squares of the monic polynomials are easy to calculate.

### 1.3 Measures and $k$ -point intensity functions

The partition function discussed above will allow us to probe global properties of the gas but not local behavior. To tackle this we will in this section develop the *one-point intensity function* which will give us the density of particles at a point. It will however take some effort but along the way we will encounter other powerful tools.

*Remark.* While the discussion in this section by no means contain any original results, its purpose is to present these standard arguments and theorems in a self contained way and in one single place. The interested reader is referred to [24, 32, 34] for alternative presentations.

### 1.3.1 Auxiliary measures

The Gibbs measure from (1.5) does not take into account the fact that our particles are identical and we are therefore doing some undercounting. To counteract this we first introduce the *marginal probability measure*  $\mathbb{P}_{n,k}$  on  $\mathbb{C}^k$  as

$$\mathbb{P}_{n,k}(\omega) = \mathbb{P}_n(\omega \times \mathbb{C}^{n-k})$$

for  $k \leq n$  and a Borel set  $\omega \subseteq \mathbb{C}^k$ . This is used to define the *correlation measure*  $\mu_{n,k}$  as

$$\mu_{n,k} = \frac{n!}{(n-k)!} \mathbb{P}_{n,k} \quad (1.12)$$

where we recognize that  $\frac{n!}{(n-k)!} = P(n, k)$  is the number of permutations of  $k$  elements in a set of size  $n$ .

*Remark.*  $\mu_{n,k}$  is not a probability measure but rather just a positive measure with total mass  $\mu_{n,k}(\mathbb{C}^k) = \frac{n!}{(n-k)!}$ . It is still useful however since we are more interested in answering questions such as “*what is the probability that there are  $k$  particles in this set?*” rather than “*what is the probability that these specific  $k$  particles are in this set?*” for which this measure is better suited.

We are now ready to handle functions which depend on the configuration of the particles. With the correlation measure we can define a sort of expectation value of a function  $X = X(\eta_1, \dots, \eta_k)$  as

$$\langle X \rangle = \int_{\mathbb{C}^k} X(\eta_1, \dots, \eta_k) d\mu_{n,k}(\eta_1, \dots, \eta_k).$$

In particular, for a fixed  $\zeta$ , we consider the function  $\chi_\varepsilon : \mathbb{C} \rightarrow \mathbb{R}$  defined as

$$\chi_\varepsilon(\eta) = \frac{1}{\varepsilon^2} \mathbf{1}_{\mathbb{D}(\zeta, \varepsilon)}(\eta), \quad \langle \chi_\varepsilon \rangle = \int_{\mathbb{C}} \chi_\varepsilon(\eta) d\mu_{n,1}(\eta) = \frac{1}{\varepsilon^2} \int_{\mathbb{D}(\zeta, \varepsilon)} d\mu_{n,1}(\eta) = \frac{\mu_{n,1}(\mathbb{D}(\zeta, \varepsilon))}{\varepsilon^2}.$$

Thus  $\langle \chi_\varepsilon \rangle$  is the expected number of particles in the disk  $\mathbb{D}(\zeta, \varepsilon)$ , per unit area. We remind the reader that with our reduced area element,  $A(\mathbb{D}(\zeta, \varepsilon)) = \varepsilon^2$ . We now let  $\varepsilon \rightarrow 0$  so that we can speak of the density at a point  $\zeta$ . We define this important quantity as the *one-point intensity function* and write it as

$$\mathbf{R}_n(\zeta) = \mathbf{R}_{n,1}(\zeta) = \lim_{\varepsilon \rightarrow 0} \langle \chi_\varepsilon \rangle = \lim_{\varepsilon \rightarrow 0} \frac{\mu_{n,1}(\mathbb{D}(\zeta, \varepsilon))}{\varepsilon^2}.$$

This type of density is often referred to as the *Janossy density* of the point process in the literature. We use boldface to indicate that the one-point intensity function is not rescaled. This will be discussed further in Section 1.3.3.

Our next goal is to generalize this to multiple points which will be known as the  *$k$ -point intensity function*. We fix  $k$  points  $\zeta_1, \dots, \zeta_k$  and define

$$X_k(\eta_1, \dots, \eta_k) = \chi_\varepsilon^{\zeta_1}(\eta_1) \cdots \chi_\varepsilon^{\zeta_k}(\eta_k), \quad \chi_\varepsilon^{\zeta}(\eta) = \frac{1}{\varepsilon^2} \mathbf{1}_{\mathbb{D}(\zeta, \varepsilon)}(\eta).$$

This appropriately generalizes the earlier  $\chi_\varepsilon$  in that  $X_k \neq 0$  if and only if  $\zeta_j$  is near  $\eta_j$  for  $j = 1, \dots, k$ . We can now define

$$\mathbf{R}_{n,k}(\zeta_1, \dots, \zeta_k) = \lim_{\varepsilon \rightarrow 0} \langle X_k \rangle = \lim_{\varepsilon \rightarrow 0} \frac{\mu_{n,k}(\mathbb{D}(\zeta_1, \varepsilon) \times \dots \times \mathbb{D}(\zeta_k, \varepsilon))}{\varepsilon^{2k}}. \quad (1.13)$$

This rather unwieldy expression is of interest to calculate. We develop some further machinery and then present a theorem to simplify it.

### 1.3.2 The correlation kernel

Let  $d\mu_n = e^{-nQ} dA$  (not to be confused with the correlation measure  $\mu_{n,k}$ ) and  $\mathcal{W}_n \subset L^2(\mathbb{C}, \mu_n)$  be the  $n$  dimensional Hilbert space consisting of all square-integrable weighted polynomials with respect to  $\mu_n$  of degree strictly less than  $n$ . Then all  $w \in \mathcal{W}_n$  are of the form  $w(\zeta) = p(\zeta)e^{-nQ(\zeta)/2}$  where  $p$  is a polynomial and  $w_1\overline{w_2} dA = p_1\overline{p_2} d\mu_n$ . We endow this space with the usual inner product

$$\langle w_1, w_2 \rangle = \int_{\mathbb{C}} w_1(\zeta)\overline{w_2(\zeta)} dA(\zeta) = \int_{\mathbb{C}} p_1(\zeta)\overline{p_2(\zeta)} d\mu_n(\zeta)$$

for  $w_1, w_2 \in \mathcal{W}_n$  and  $p_1, p_2$  the corresponding polynomials. Let  $w_0, \dots, w_{n-1}$  denote an orthonormal basis in this space as can be obtained by for example the Gram-Schmidt process. These orthogonal polynomials will allow us to determine various properties of the space. We define the *correlation kernel*  $\mathbf{K}_n$  as

$$\mathbf{K}_n(\zeta, \eta) = w_0(\zeta)\overline{w_0(\eta)} + \dots + w_{n-1}(\zeta)\overline{w_{n-1}(\eta)} = \sum_{j=0}^{n-1} p_j(\zeta)\overline{p_j(\eta)} e^{-nQ(\zeta)/2} e^{-nQ(\eta)/2}$$

where we note that  $\mathbf{K}_n(\zeta, \zeta) = \sum_{j=0}^{n-1} |w_j(\zeta)|^2$ . The correlation kernel can be related to the  $k$ -point intensity function via the following theorem. Its proof is inspired by that in [24].

**Theorem 1.3.1.** *Let  $\mathbf{R}_{n,k}$  denote the  $k$ -point intensity function and  $\mathbf{K}_n$  the correlation kernel. Then*

$$\mathbf{R}_{n,k}(\zeta_1, \dots, \zeta_k) = \det[\mathbf{K}_n(\zeta_i, \zeta_j)]_{i,j=1}^k. \quad (1.14)$$

*Remark.* Point processes with intensity functions of the above form are said to be *determinantal* point processes. Note again that  $\{\zeta_j\}_{j=1}^n$  is only determinantal in the  $\beta = 1$  case which is why we were bound to that value for the inverse temperature.

In the case of  $k = 1$  we have the following immediate corollary.

**Corollary 1.3.2.** *The one-point intensity function  $\mathbf{R}_n(\zeta)$  takes the form*

$$\mathbf{R}_n(\zeta) = \mathbf{K}_n(\zeta, \zeta) = \sum_{j=0}^{n-1} |w_j(\zeta)|^2 = \sum_{j=0}^{n-1} |p_j(\zeta)|^2 e^{-nQ(\zeta)}.$$

This explicit formula can be used to calculate the one-point intensity function numerically or by hand by deriving implicit or explicit expressions for the orthogonal polynomials. Before attempting a proof we present and prove three helpful lemmas. This first lemma detail some additional properties of the correlation kernel.

**Lemma 1.3.3** ([34, p. 251]). *For the correlation kernel  $\mathbf{K}_n(\zeta, \eta) = w_0(\zeta)\overline{w_0(\eta)} + \dots + w_{n-1}(\zeta)\overline{w_{n-1}(\eta)}$  the following properties hold.*

$$\begin{aligned} (i) \quad & \int_{\mathbb{C}} \mathbf{K}_n(t, \xi)\mathbf{K}_n(\xi, s) dA(\xi) = \mathbf{K}_n(t, s), & (iii) \quad & \int_{\mathbb{C}} \mathbf{K}_n(t, t) dA(t) = n, \\ (ii) \quad & \iint_{\mathbb{C}^2} |\mathbf{K}_n(t, s)|^2 dA(t) dA(s) = n, & (iv) \quad & \langle q, \mathbf{K}_n \rangle = q \text{ for all } q \in \mathcal{W}_n. \end{aligned}$$

*Proof.* For (i), the result follows from a straight forward calculation.

$$\int_{\mathbb{C}} \mathbf{K}_n(t, \xi) \mathbf{K}_n(\xi, s) dA(\xi) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} w_i(t) w_j(s) \int_{\mathbb{C}} \overline{w_i(\xi)} w_j(\xi) dA(\xi) = \mathbf{K}_n(t, s).$$

In proving (ii), we bunch together terms and integrate using the orthogonality relations.

$$\begin{aligned} \iint_{\mathbb{C}^2} |\mathbf{K}_n(t, s)|^2 dA(t) dA(s) &= \sum_{i=0}^{n-1} \int_{\mathbb{C}} \overline{w_i(s)} w_j(s) \sum_{j=0}^{n-1} \int_{\mathbb{C}} w_i(t) \overline{w_j(t)} dA(t) dA(s) \\ &= \sum_{i=0}^{n-1} \int_{\mathbb{C}} \overline{w_i(s)} w_i(s) dA(s) = n \end{aligned}$$

where we in the second equality used that only terms where  $i = j$  survives. For (iii) we need only change the order of integration and summation.

$$\int_{\mathbb{C}} \mathbf{K}_n(t, t) dA(t) = \int_{\mathbb{C}} \sum_{i=0}^{n-1} w_i(t) \overline{w_i(t)} dA(t) = \sum_{i=0}^{n-1} \int_{\mathbb{C}} w_i(t) \overline{w_i(t)} dA(t) = n.$$

Lastly for (iv), let  $q(\zeta) = \sum_{j=0}^{n-1} a_j w_j(\zeta)$ . Then

$$\langle q(\zeta), \mathbf{K}_n(\zeta, \eta) \rangle = \int_{\mathbb{C}} \sum_{i=0}^{n-1} a_i w_i(\zeta) \sum_{j=0}^{n-1} \overline{w_j(\zeta) w_j(\eta)} dA(\zeta) = \sum_{i=0}^{n-1} a_i w_i(\eta) = q(\eta).$$

□

*Remark.* Kernels for which property (iv) is valid are said to be *reproducing kernels*. The correlation kernel is uniquely determined by it being a reproducing kernel as can be seen by noting that if  $\mathbf{K}$  and  $\mathbf{K}'$  are reproducing kernels, then

$$\langle q, \mathbf{K} - \mathbf{K}' \rangle = q - q = 0 \quad \text{for all } q \in \mathcal{W}_n.$$

This implies that  $\mathbf{K} = \mathbf{K}'$ . See [15] and [7] for more on spaces such as  $\mathcal{W}_n$  and their reproducing kernels.

The following lemma will be crucial in proving Theorem 1.3.1. The proof is based on that of Theorem 5.1.2 in [31, p. 89] and only makes use of some the properties detailed in the above lemma. It could be generalized to other kernels but we will have no need for it so we state and prove this special case.

**Lemma 1.3.4.** *Let  $\mathbf{K}_n$  denote the correlation kernel of  $\mathcal{W}_n$ . Then*

$$\int_{\mathbb{C}} \det[\mathbf{K}_n(\zeta_i, \zeta_j)]_{i,j=1}^k dA(\zeta_k) = (n - k + 1) \det[\mathbf{K}_n(\zeta_i, \zeta_j)]_{i,j=1}^{k-1}.$$

*Proof.* We again use the Leibniz formula for the determinant. Let  $i \in S_n$  denote a permutation such that  $i : k \mapsto i_k$  and  $\sigma(i)$  the sign function of permutations taking the values +1 and -1 when the permutation  $i$  is even and odd respectively. Then

$$\int_{\mathbb{C}} \det[\mathbf{K}_n(\zeta_i, \zeta_j)]_{i,j=1}^k dA(\zeta_k) = \sum_{i \in S_n} \sigma(i) \int_{\mathbb{C}} \prod_{\ell=1}^k \mathbf{K}_n(\zeta_{i_\ell}, \zeta_\ell) dA(\zeta_k).$$



Integrating this we are looking for cancellations by means of (i) and (iii) of Lemma 1.3.3. We separate the permutation into two types; the  $(k-1)! \cdot 1$  terms for which  $i_k = k$  and  $(k-1)! \cdot (k-1)$  terms for which  $i_k \neq k$ . For the first type, integration will give a factor  $n$  by property (iii) and the remainder of the permutation will have the same parity since the factor we've removed has parity 1 since  $i : k \mapsto k$ . We can thus write

$$\int_{\mathcal{C}} \det[\mathbf{K}_n(\zeta_i, \zeta_j)]_{i,j=1}^k dA(\zeta_k) = n \cdot \det[\mathbf{K}_n(\zeta_i, \zeta_j)]_{i,j=1}^{k-1} + \sum_{\substack{i \in S_n, \\ i_k \neq k}} \sigma(i) \int_{\mathcal{C}} \prod_{\ell=1}^k \mathbf{K}_n(\zeta_{i_\ell}, \zeta_\ell) dA(\zeta_k).$$

For the  $(k-1)! \cdot (k-1)$  terms for which  $i_k \neq k$  there will be a factor of the form  $\mathbf{K}_n(t, \xi) \mathbf{K}_n(\xi, s)$  which will collapse to  $\mathbf{K}_n(t, s)$  upon integration by property (i). The remaining product will have  $k-1$  terms and reversed parity since  $i_k \neq k$ . In summary

$$\int_{\mathcal{C}} \det[\mathbf{K}_n(\zeta_i, \zeta_j)]_{i,j=1}^k dA(\zeta_k) = n \cdot \det[\mathbf{K}_n(\zeta_i, \zeta_j)]_{i,j=1}^{k-1} - (k-1) \cdot \det[\mathbf{K}_n(\zeta_i, \zeta_j)]_{i,j=1}^{k-1}$$

which is what we wished to show.  $\square$

We now prove a special case of Theorem 1.3.1, namely  $k = n$ , which will be the base case when we prove the general result using induction.

**Lemma 1.3.5.** *Let  $\mathbf{R}_{n,n}$  denote the  $n$ -point intensity function and  $\mathbf{K}_n$  the correlation kernel. Then*

$$\mathbf{R}_{n,n}(\zeta_1, \dots, \zeta_n) = \det[\mathbf{K}_n(\zeta_i, \zeta_j)]_{i,j=1}^n.$$

*Proof.* We begin by rewriting (1.8) so that it is expressed in regular weighted orthogonal polynomials instead of monic unweighted ones. This is easy as  $w_j \propto \pi_j, \|w_j\| = 1 \implies w_j = \frac{\pi_j}{\|\pi_j\|} e^{-nQ/2}$ . We will recover  $\det[\mathbf{K}_n(\zeta_i, \zeta_j)]_{i,j=1}^n$  by considering  $|\det V_n|^2$  as the determinant of a product of matrices. Recall that for any square matrix  $A$ ,  $|\det A|^2 = \det(A^* \cdot A) = \det(A^T \cdot \bar{A})$ . Setting  $A = V_n$  we find

$$\begin{aligned} |\det V_n|^2 &= \det \left( \begin{bmatrix} \pi_0(\zeta_1) & \pi_1(\zeta_1) & \cdots & \pi_{n-1}(\zeta_1) \\ \pi_0(\zeta_2) & \pi_1(\zeta_2) & \cdots & \pi_{n-1}(\zeta_2) \\ \vdots & \vdots & \ddots & \vdots \\ \pi_0(\zeta_n) & \pi_1(\zeta_n) & \cdots & \pi_{n-1}(\zeta_n) \end{bmatrix} \cdot \overline{\begin{bmatrix} \pi_0(\zeta_1) & \pi_0(\zeta_2) & \cdots & \pi_0(\zeta_n) \\ \pi_1(\zeta_1) & \pi_1(\zeta_2) & \cdots & \pi_1(\zeta_n) \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{n-1}(\zeta_1) & \pi_{n-1}(\zeta_2) & \cdots & \pi_{n-1}(\zeta_n) \end{bmatrix}} \right) \\ &= \underbrace{\frac{\|\pi_0\|^2 \cdots \|\pi_{n-1}\|^2}{e^{-n(Q(\zeta_1) + \dots + Q(\zeta_n))}}}_{\text{Scaling } \pi_j \rightarrow w_j} \left| \begin{array}{cccc} \sum_{i=0}^{n-1} |w_i(\zeta_1)|^2 & \sum_{i=0}^{n-1} w_i(\zeta_1) \overline{w_i(\zeta_2)} & \cdots & \sum_{i=0}^{n-1} w_i(\zeta_1) \overline{w_i(\zeta_n)} \\ \sum_{i=0}^{n-1} w_i(\zeta_2) \overline{w_i(\zeta_1)} & \sum_{i=0}^{n-1} |w_i(\zeta_2)|^2 & \cdots & \sum_{i=0}^{n-1} w_i(\zeta_2) \overline{w_i(\zeta_n)} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=0}^{n-1} w_i(\zeta_n) \overline{w_i(\zeta_1)} & \sum_{i=0}^{n-1} w_i(\zeta_n) \overline{w_i(\zeta_2)} & \cdots & \sum_{i=0}^{n-1} |w_i(\zeta_n)|^2 \end{array} \right| \\ &= \frac{Z_n}{n!} \det[\mathbf{K}_n(\zeta_i, \zeta_j)]_{i,j=1}^n e^{n(Q(\zeta_1) + \dots + Q(\zeta_n))} \end{aligned}$$

where we used Theorem 1.2.4 for the last step. We expand  $\mathbf{R}_{n,n}$  using (1.13) and (1.12) as

$$\mathbf{R}_{n,n}(\zeta_1, \dots, \zeta_n) = \lim_{\varepsilon \rightarrow 0} \frac{n! \mathbb{P}_n(\mathbb{D}(\zeta_1, \varepsilon) \times \dots \times \mathbb{D}(\zeta_n, \varepsilon))}{0! \varepsilon^{2n}} = \frac{n!}{Z_n} |\det V_n|^2 \cdot e^{-n(Q(\zeta_1) + \dots + Q(\zeta_n))}$$

where the last equality follows from the continuity of  $\mathbb{P}_n$ . The desired result follows upon comparison of the two above equations.  $\square$

With this we are able to complete the proof of Theorem 1.3.1.

*Proof of Theorem 1.3.1.* We use induction on  $k$  with  $k$  decreasing and  $k = n$  as the base case. Suppose that (1.14) holds for  $k$ , i.e.

$$\mathbf{R}_{n,k} = \frac{n!}{(n-k)!} e^{-n(Q(\zeta_1)+\dots+Q(\zeta_k))} \int_{\mathbb{C}^{n-k}} |\det V_n|^2 dA(\zeta_{k+1}) \cdots dA(\zeta_n) = \det[\mathbf{K}_n(\zeta_i, \zeta_j)]_{i,j=1}^k.$$

We can now treat the  $k-1$  case as

$$\begin{aligned} \mathbf{R}_{n,k-1} &= \frac{n!}{(n-k+1)!} e^{-n(Q(\zeta_1)+\dots+Q(\zeta_{k-1}))} \int_{\mathbb{C}} \underbrace{\int_{\mathbb{C}^{n-k}} |\det V_n|^2 dA(\zeta_{k+1}) \cdots dA(\zeta_n)}_{=\frac{(n-k)!}{n!} e^{n(Q(\zeta_1)+\dots+Q(\zeta_k))} \det[\mathbf{K}_n(\zeta_i, \zeta_j)]_{i,j=1}^k} dA(\zeta_k) \\ &= \frac{1}{n-k+1} \int_{\mathbb{C}} \det[\mathbf{K}_n(\zeta_i, \zeta_j)]_{i,j=1}^k dA(\zeta_k) = \det[\mathbf{K}_n(\zeta_i, \zeta_j)]_{i,j=1}^{k-1}. \end{aligned}$$

where we used Lemma 1.3.4 for last step. We conclude that equality holds for  $k = 1, \dots, n$  and so the theorem is proved.  $\square$

### 1.3.3 Rescaling correlation kernels

Since we are for the most part interested in the behavior of our system as  $n \rightarrow \infty$ , we need to zoom at a scale dependent on  $n$  in order for certain quantities to converge. When zooming into a point  $p$  a natural scale to rescale at is the scale  $r_n$  at which

$$n \int_{\mathbb{D}(p, r_n)} \Delta Q dA = 1 \implies r_n \sim \frac{1}{\sqrt{n \Delta Q(p)}} \text{ as } n \rightarrow \infty. \quad (1.15)$$

This is because  $\Delta Q$  is the limiting macroscopic density of particles which is discussed further in the Section 1.4. For now we may take it for granted. If  $\Delta Q(p) = 0$  we say that  $p$  is a *singular point* and we need to calculate the above integral. We define the rescaled point process  $\{z_j\}_{j=1}^n$  around the point  $p$  by

$$z_j = r_n^{-1}(\zeta_j - p).$$

We determine how the correlation kernel is rescaled by determining the orthogonal polynomials in the rescaled system. Making the ansatz  $\tilde{w} = C \cdot w$  where  $\tilde{w}$  and  $w$  are orthogonal polynomials in the rescaled system and the original system respectively we have that  $dz = r_n^{-1} d\zeta$  and

$$1 = \langle \tilde{w}, \tilde{w} \rangle = \int_{\mathbb{C}} \tilde{w}(z) \overline{\tilde{w}(z)} e^{-nQ(z)} = C^2 r_n^{-2} \langle w, w \rangle.$$

We conclude that  $C = r_n$  and taking non-boldface characters to mean non-rescaled quantities we have

$$K_n(z, w) = r_n^2 \mathbf{K}_n(\zeta, \eta) \implies R_{n,k}(z_1, \dots, z_k) = r_n^{2k} \mathbf{R}_{n,k}(\zeta_1, \dots, \zeta_k). \quad (1.16)$$

We can state this as  $K_n$  is the correlation kernel of  $\{z_j\}_{j=1}^n$ . For more on rescaling, see e.g. [5].

## 1.4 The equilibrium measure

In this section we discuss a general result on the distribution of  $\{\zeta_i\}_{i=1}^n$  as  $n \rightarrow \infty$ . For proofs and a more in-depth discussion we refer the reader to [34]. Let once again  $Q : \mathbb{C} \rightarrow \mathbb{R} \cup \{+\infty\}$  and impose the conditions that  $Q$  is lower semi-continuous, real analytic and  $\liminf_{\zeta \rightarrow \infty} \frac{Q(\zeta)}{\log|\zeta|^2} > 1$ . Such a potential is said to be *admissible*. For a Borel probability measure  $\mu$  on  $\mathbb{C}$  and such an admissible potential  $Q$ , we define the weighted logarithmic energy  $I_Q$  as

$$I_Q(\mu) = \iint \log \frac{1}{|\zeta - \eta|} d\mu(\zeta) d\mu(\eta) + \mu(Q).$$

By a theorem of Frostman, there exists a measure  $\sigma$ , known as the *equilibrium measure*, which minimizes this quantity among compactly supported probability measures. We denote the compact support of  $\sigma$  by  $S = \text{supp } \sigma$  and call it the *droplet*. The theorem further states that  $\sigma$  takes the form

$$d\sigma = \Delta Q \cdot \mathbf{1}_S dA$$

and  $\Delta Q \geq 0$  on  $S$ . The equilibrium measure is of interest because it is the limiting behavior of  $\{\zeta_i\}_{i=1}^n$  in the sense that for a bounded continuous function  $f$ ,

$$\frac{1}{n} \mathbb{E}[f(\zeta_1) + \dots + f(\zeta_n)] \rightarrow \sigma(f) \quad \text{as } n \rightarrow \infty$$

where  $\mathbb{E}$  is the expectation value with respect to  $\mathbb{P}_n$ . The interpretation is that as  $n$  increases, particles will tend to occupy only the droplet with density proportional to  $\Delta Q$ . As one might guess,  $\Delta Q$  is intimately connected to the one-point intensity function and in fact  $n\Delta Q$  is the leading order contribution to  $\mathbf{R}$  where the factor  $n$  is a consequence of  $\sigma$  being a probability measure [25].

For a compact  $\Sigma \subset \mathbb{C}$  and a potential  $Q$  we say that the energy  $V$  of  $\Sigma$  is

$$V = \inf \{I_Q(\mu), \mu \text{ is a Borel probability measure with } \text{supp } \mu = \Sigma\}$$

and write  $\text{cap}(\Sigma) = e^{-V}$  for the *capacity* of  $\Sigma$ . A property that holds on  $\mathbb{C} \setminus \Sigma$  for some  $\Sigma$  with  $\text{cap}(\Sigma) = 0$  is said to hold *quasi-everywhere* (q.e.).

### 1.4.1 An alternative characterization of the droplet

The *logarithmic potential*  $U^\sigma$  associated to the equilibrium measure  $\sigma$  is given by

$$U^\sigma(\zeta) = \int_{\mathbb{C}} \log \frac{1}{|\zeta - t|} d\sigma(t) = \int_S \Delta Q \log \frac{1}{|\zeta - t|} dA(t).$$

As hinted by the terminology, this potential will tend to agree with the external potential  $Q$ . Indeed, it turns out that the quantity  $Q + 2U^\sigma$  assumes a constant minimum  $\gamma$  on  $S$  where  $\gamma$  is known as *Robin's constant* while outside of  $S$  we have  $Q + 2U^\sigma > \gamma$ . We can form  $\hat{Q}(\zeta) = 2U^\sigma(\zeta) - \gamma$  which clearly has the properties  $\hat{Q} = Q$  on  $S$  and  $\hat{Q} < Q$  on  $\mathbb{C} \setminus S$ . This next theorem gives a characterization of  $\hat{Q}$  which will allow us to determine  $S$ .

**Theorem 1.4.1** ([34, Theorem I.4.1]). *Let SH be the set of all subharmonic function  $f : \mathbb{C} \rightarrow \mathbb{R}$  such that  $f(\zeta) \leq Q(\zeta)$  for all  $\zeta$  and  $f(\zeta) - \log|\zeta|^2$  is bounded as  $\zeta \rightarrow \infty$ . Then*

$$\hat{Q}(\zeta) = \sup_{f \in \text{SH}} f(\zeta), \quad \text{q.e.}$$

**Example 1.4.2.** Consider the case of  $Q(\zeta) = |\zeta|^2 - \log |\zeta|^2$  which can be seen as the Ginibre ensemble with a growing repulsive charge of strength  $n$  at 0. Since this potential is radially symmetric we can expect the droplet to take the form of an annulus. Calculating  $\Delta Q$  is not enough to determine  $S$  as

$$\Delta Q(\zeta) = 1 - 2\Delta \log |z| = 1 \quad \text{for } \zeta \in \mathbb{C} \setminus \{0\}$$

is the same density as for  $Q(\zeta) = |\zeta|^2$  apart from at  $\{0\}$ . We instead make use of Theorem 1.4.1. When forming  $\hat{Q}$  we first consider only the radial harmonic functions  $f(r) = a + b \log r$  where  $b \in [0, 2]$  from the growth condition in the theorem. Forming  $D(r) = Q(r) - f(r) = r^2 - a - (2+b) \log r$  we note that the  $f \leq Q$  condition is equivalent to  $D \geq 0$ . Fixing some  $r$ , we know that  $r \in S$  if we can fix  $a, b$  such that  $D(r) = 0$ . Noting that

$$0 = D'(r) = 2r - \frac{2+b}{r} \implies r = \pm \sqrt{\frac{2+b}{2}}$$

we pick an appropriate  $a$  such that this minimum is reached at a point where  $D(r) = 0$ . From the above condition on  $r$  we also see that  $b \in [0, 2]$  corresponds to  $r \in [1, \sqrt{2}]$ . The question is now if there are any function in SH such that  $Q(r) = \hat{Q}(r)$  for points outside  $r \in [1, \sqrt{2}]$ . This cannot be the case as that would imply that  $\int_S d\sigma > 1$  since  $\int_{1 \leq r \leq \sqrt{2}} d\sigma = 1$  and so  $S = \{\zeta \in \mathbb{C}, 1 \leq |\zeta| \leq \sqrt{2}\}$  which we remark is an example of a droplet which can not be written as  $S = \{\zeta \in \mathbb{C}, Q(\zeta) \leq A\}$  for some  $A$ .

## Chapter 2

# Correlation kernel of the Mittag-Leffler ensemble

Now that we have developed the theoretical machinery required to calculate the density of particles at a point we put it to use by considering some examples. Most of the material in this chapter has been discussed elsewhere such as [4, 6] and we reproduce and clarify those discussions somewhat. We also remark that for the boundary case, the result is just a case of the universal erfc kernel obtained in [26].

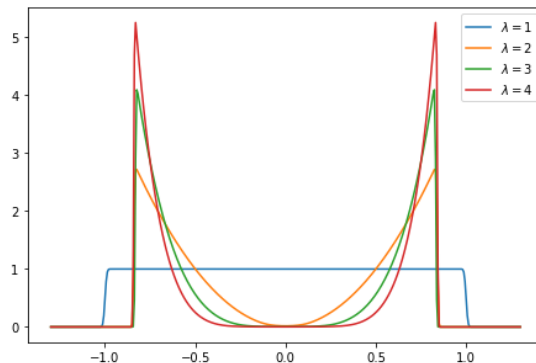


Figure 2.1: Unscaled numerical  $R_n(z)$  for a collection of Mittag-Leffler ensembles.

We refer to potentials of the form

$$Q(\zeta) = |\zeta|^{2\lambda} - \frac{2c}{n} \log |\zeta|, \quad \lambda \geq 0, c > -1$$

as *Mittag-Leffler potentials* and ensembles picked with respect to the induced probability measure as *Mittag-Leffler ensembles*. The example in Figure 1.1 correspond to the special case  $\lambda = 1, c = 0$  which is known as the *Ginibre ensemble* for historical reasons. The  $|\zeta|^{2\lambda}$  term is just a sufficiently aggressive confining potential (with higher  $\lambda$  corresponding to more aggressive scaling) while the  $\frac{2c}{n} \log |\zeta|$  term is equivalent to a fixed inserted point charge of strength  $c$  at 0. Indeed,  $e^{-n(\frac{2c}{n} \log |\zeta|)} = |\zeta - 0|^{2c}$  and in view of (1.5) this can be seen as adding a point  $\zeta_{n+1}$  at 0 which is unaffected by the external potential,  $c$  times.

In this chapter we will calculate the correlation kernel of the Mittag-Leffler ensemble in the center of the droplet and on its boundary. However for or the boundary case we will only be able to calculate the correlation kernel of the Ginibre case but we will examine the situation numerically for higher  $\lambda$ .

We first determine some properties of the ensembles which will be needed when calculating the correlation kernels.

## 2.1 Ensemble properties

### 2.1.1 Determining the droplet

In order to calculate the correlation kernel at the boundary of the droplet we need to know what the droplet looks like. To do so we will make use of the discussion in Section 1.4, more specifically the fact that  $\sigma, d\sigma = \Delta Q \cdot \mathbf{1}_S dA$  is a probability measure which implies that  $\int_{\mathbb{C}} d\sigma = 1$ . Since  $Q$  is radially symmetric we can assume that the same is true for  $S$ . Since the inserted point charge at 0 is a microscopic perturbation and we are interested in the macroscopic behavior we can discard it and conclude that the droplet is a disk and not an annulus since  $|\zeta|^{2\lambda}$  has its minimum at  $\zeta = 0$ . Letting  $R$  denote the radius we have

$$1 = \int_S \Delta Q(\zeta) dA(\zeta) \lambda^2 \int_S |\zeta|^{2\lambda-2} dA(\zeta) = \frac{\lambda^2}{\pi} \int_0^{2\pi} d\theta \int_0^R r^{2\lambda-1} dr = 2\lambda^2 \frac{R^{2\lambda}}{2\lambda} = \lambda R^{2\lambda}$$

yielding  $S = \{\zeta \in \mathbb{C}, |\zeta| \leq \lambda^{-\frac{1}{2\lambda}}\}$ . We plot the radius below for reference.

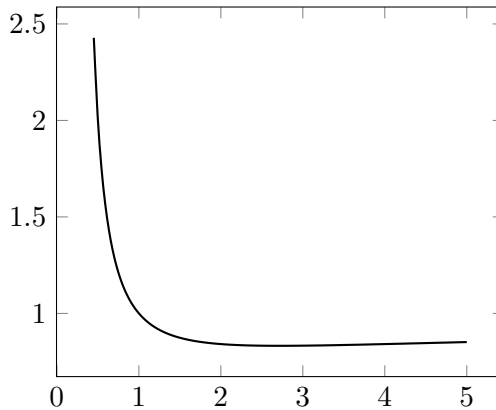


Figure 2.2: Plot of Mittag-Leffler ensemble radius  $\lambda^{-\frac{1}{2\lambda}}$ .

For  $\lambda \geq 1$ ,  $\lambda^{-\frac{1}{2\lambda}} \leq 1$  with equality only at  $\lambda = 1$ . The radius attains its minimum of  $\approx 0.83$  at  $\lambda = e$  and  $\lim_{\lambda \rightarrow \infty} \lambda^{-\frac{1}{2\lambda}} = 1$ .

*Remark.* The different radii can be seen in practice by comparing Figures 1.2 and 1.3.

### 2.1.2 Scaling factor

#### Interior

For the interior correlation kernel we consider the point  $p = 0$ . Recalling the discussion in Section 1.3.3 we begin by calculating  $\Delta Q$ . We can discard the  $\log |\zeta|$  term to avoid singularities at the origin, partly because of the built in  $\frac{1}{n}$  scaling and partly because we are more interested in the macroscopic behavior of the system when rescaling. We then have that  $\Delta Q(\zeta) = 2\lambda |\zeta|^{2\lambda-2}$ . Since  $\Delta Q(0) = 0$  for  $\lambda > 1$  we consider the integral in

(1.15). Doing so we obtain

$$n \int_{\mathbb{D}(p, r_n)} \Delta Q dA = 1 = 2\lambda 2n \int_0^{r_n} r^{2\lambda-1} dr = 2nr_n^{2\lambda} \implies r_n \sim \left(\frac{1}{n}\right)^{\frac{1}{2\lambda}}$$

where we take  $\sim$  to indicate the asymptotic dependence on  $n$  meaning we discarded a factor  $\frac{1}{2^{2\lambda}}$  for simplicity.

### Boundary

The boundary point  $\zeta = \lambda^{\frac{-1}{2\lambda}} \in \partial S$  is not singular so we can calculate it directly using the second expression in (1.15) as

$$r_n \sim \frac{1}{\sqrt{n\Delta Q\left(\lambda^{\frac{-1}{2\lambda}}\right)}} = \frac{1}{\sqrt{n2\lambda^{\frac{1}{\lambda}}}} \sim \frac{1}{\sqrt{n}}.$$

### 2.1.3 Orthogonal polynomials

Generally when calculating correlation kernels, the main difficulty lies in determining the orthogonal polynomials. The radial symmetry of the Mittag-Leffler potential simplifies the problem by having the orthogonal polynomials be monomials as we will see in this next lemma.

**Lemma 2.1.1.** *For  $Q(\zeta) = |\zeta|^{2\lambda} - \frac{2c}{n} \log |\zeta|$  we have*

$$\langle \zeta^j, \zeta^k \rangle = \delta_{j,k} \frac{\Gamma\left(\frac{k+c+1}{\lambda}\right)}{\lambda n^{\frac{k+c+1}{\lambda}}}.$$

*Proof.* The integral takes the form

$$\int_{\mathbb{C}} |\zeta|^{2c} \zeta^j \bar{\zeta}^k e^{-n|\zeta|^{2\lambda}} dA(\zeta) = \delta_{j,k} 2 \int_0^\infty r^{2k+2c+1} e^{-nr^{2\lambda}} dr.$$

Making the substitution  $t = nr^{2\lambda}$  yields

$$\delta_{j,k} \frac{1}{\lambda n} \int_0^\infty \left(\frac{t}{n}\right)^{\frac{k+c-\lambda+1}{\lambda}} e^{-t} dt = \delta_{j,k} \frac{\Gamma\left(\frac{k+c+1}{\lambda}\right)}{\lambda n^{\frac{k+c+1}{\lambda}}}$$

which is what we wanted to show.  $\square$

By the above lemma it is clear that the scalar product of the monomials of different degrees is zero and so the monomials are the orthogonal polynomials. Normalizing appropriately we have

$$p_k(\zeta) = \zeta^k \sqrt{\frac{\lambda n^{\frac{k+c+1}{\lambda}}}{\Gamma\left(\frac{k+c+1}{\lambda}\right)}} \implies p_k(\zeta) \overline{p_k(\eta)} = \zeta^k \bar{\eta}^k \frac{\lambda n^{\frac{k+c+1}{\lambda}}}{\Gamma\left(\frac{k+c+1}{\lambda}\right)}.$$

## 2.2 Interior correlation kernel

We first discuss the correlation kernel at  $\zeta = 0$ . From the scaling factor discussion above we have the scaling relations

$$\zeta = \frac{z}{n^{2\lambda}}, \quad \eta = \frac{w}{n^{2\lambda}}.$$

Substituting this into (1.16) we have

$$\begin{aligned} K_n(z, w) &= \left(\frac{1}{n}\right)^{\frac{1}{\lambda}} \sum_{k=0}^{n-1} \frac{\lambda n^{\frac{k+c+1}{\lambda}}}{\Gamma(\frac{k+c+1}{\lambda})} \left(\frac{z}{n^{2\lambda}}\right)^k \left(\frac{\bar{w}}{n^{2\lambda}}\right)^k e^{-\frac{1}{2}n\left|\frac{z}{n^{1/2\lambda}}\right|^{2\lambda} - \frac{1}{2}n\left|\frac{w}{n^{1/2\lambda}}\right|^{2\lambda} + c \log\left|\frac{z}{n^{1/2\lambda}}\right| + c \log\left|\frac{w}{n^{1/2\lambda}}\right|} \\ &= \lambda |zw|^c \sum_{k=0}^{n-1} \frac{(z\bar{w})^k}{\Gamma(\frac{k+c+1}{\lambda})} e^{-\frac{1}{2}|z|^{2\lambda} - \frac{1}{2}|w|^{2\lambda}} \xrightarrow{n \rightarrow \infty} \lambda |zw|^c E_{\frac{1}{\lambda}, \frac{c+1}{\lambda}}(z\bar{w}) e^{-\frac{1}{2}|z|^{2\lambda} - \frac{1}{2}|w|^{2\lambda}} \end{aligned}$$

where

$$E_{a,b}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(aj + b)}$$

is known as the *two-parametric Mittag-Leffler function*. This leads to the limiting one-point intensity function

$$R(z) = \lambda |z|^{2c} E_{\frac{1}{\lambda}, \frac{c+1}{\lambda}}(|z|^2) e^{-|z|^{2\lambda}}. \quad (2.1)$$

In the special  $\lambda = 1, c = 0$  case, the Ginibre ensemble, we get  $R \equiv 1$  which is to be expected since  $\Delta Q \equiv 1$  in a neighborhood of  $\zeta = 0$ . We can also consider the most basic case of an inserted point charge,  $\lambda = 1, c = 1$  for which we obtain

$$R(z) = |z|^2 \sum_{k=0}^{\infty} \frac{|z|^{2k}}{\Gamma(k+2)} e^{-|z|^2} = \sum_{k=0}^{\infty} \frac{|z|^{2(k+1)}}{(k+1)!} e^{-|z|^2} = 1 - e^{-|z|^2}.$$

Below we show plots for  $R(z)$  for the collection of configurations shown in the legend, including the two cases discussed above.

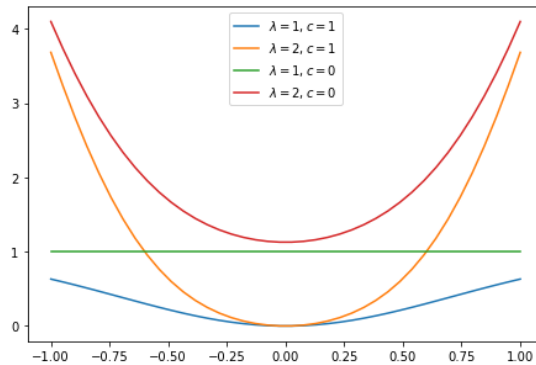


Figure 2.3: Numerical  $R(z)$  in a neighborhood of  $z = 0$  for a collection of  $\lambda$  and  $c$ 's.



## 2.3 Boundary correlation kernel

In the boundary case we have the scaling relations

$$\zeta = \lambda^{\frac{-1}{2\lambda}} + \frac{z}{\sqrt{n}}, \quad \eta = \lambda^{\frac{-1}{2\lambda}} + \frac{w}{\sqrt{n}}.$$

Without plugging in the scaling relations we can write the correlation kernel implicitly as

$$K_n(z, w) = \frac{1}{n} \sum_{k=0}^{n-1} \frac{\lambda n^{\frac{k+c+1}{\lambda}} \zeta^k \bar{\eta}^k}{\Gamma(\frac{k+c+1}{\lambda})} e^{-\frac{1}{2}n|\zeta|^{2\lambda} - \frac{1}{2}n|\eta|^{2\lambda} + c \log|\zeta| + c \log|\eta|}. \quad (2.2)$$

We will not be able to provide an analytical expression for this as in the interior case except for the Ginibre ensemble configuration ( $\lambda = 1, c = 0$ ). We can however investigate the general situation numerically which we will do in the second subsection.

### 2.3.1 The Ginibre case

The Ginibre boundary correlation kernel is a well-known result and we follow the presentation from [4]. Setting  $\lambda = 1, c = 0$  in (2.2), we find

$$K_n(z, w) = \sum_{j=0}^{n-1} \frac{(n\zeta\bar{\eta})^j}{j!} e^{-\lambda} = \sum_{j=0}^{n-1} \left( \frac{n\zeta\bar{\eta}}{\lambda} \right)^j \frac{\lambda^j}{j!} e^{-\lambda}, \quad \lambda = \frac{n}{2} (|\zeta|^2 + |\eta|^2).$$

We now introduce the Poisson distributed random variable  $X_n$  with intensity  $\lambda = \lambda(n)$ . Recall that for a random variable  $X$  and a function  $f$  the expectation value of  $f(X)$  is written as

$$\mathbb{E}[f(X)] = \sum_{j=0}^{\infty} \mathbb{P}(X = j) \cdot f(j).$$

Setting  $f(x) = \left( \frac{n\zeta\bar{\eta}}{\lambda} \right)^x$  and  $X = X_n$  we thus have

$$\mathbb{E} \left[ \left( \frac{n\zeta\bar{\eta}}{\lambda} \right)^{X_n} \cdot \mathbf{1}_{\{X_n < n\}} \right] = \sum_{j=0}^{n-1} \mathbb{P}[X_n = j] \left( \frac{n\zeta\bar{\eta}}{\lambda} \right)^j = \sum_{j=0}^{n-1} \left( \frac{n\zeta\bar{\eta}}{\lambda} \right)^j \frac{\lambda^j}{j!} e^{-\lambda} = K_n(z, w).$$

Next we introduce a new random variable  $Y_n$  with  $X_n = \lambda + \sqrt{\lambda}Y_n$ . Then

$$K_n(z, w) = \underbrace{\left( \frac{n\zeta\bar{\eta}}{\lambda} \right)^\lambda}_{=A_n} \cdot \underbrace{\mathbb{E} \left[ \left( \frac{n\zeta\bar{\eta}}{\lambda} \right)^{\sqrt{\lambda}Y_n} \cdot \mathbf{1}_{\{Y_n < \alpha_n\}} \right]}_{=B_n}. \quad (2.3)$$

Where  $X_n < n \implies \lambda + \sqrt{\lambda}Y_n < n \implies Y_n < (n - \lambda)/\sqrt{\lambda} =: \alpha_n$ . We note that  $Y_n$  has mean 0 and variance 1 and so as  $n \rightarrow \infty$  it will converge in distribution to the standard normal by the central limit theorem.

#### Proposition 2.3.1.

$$A_n = e^{ib\sqrt{n} + b^2/2} G(z, w)(1 + o(1)), \quad B_n \rightarrow e^{-b^2/2} F(z + \bar{w}) \text{ as } n \rightarrow \infty$$

where  $b = \text{Im}(z + \bar{w})$ ,

$$G(z, w) = e^{z\bar{w} - |z|^2/2 - |w|^2/2}, \quad F(z) = \frac{1}{2} \operatorname{erfc} \left( \frac{z}{\sqrt{2}} \right) = \frac{1}{\sqrt{2\pi}} \int_z^\infty e^{-\zeta^2/2} d\zeta.$$

*Proof.* First note that  $\lambda$  can be written as

$$\begin{aligned}\lambda &= \frac{n}{2} (|\zeta|^2 + |\eta|^2) = \frac{n}{2} \left( \left| 1 + \frac{z}{\sqrt{n}} \right|^2 + \left| 1 + \frac{w}{\sqrt{n}} \right|^2 \right) \\ &= \frac{n}{2} \left( 1 + \frac{2}{2\sqrt{n}} \operatorname{Re}(z) + \frac{|z|^2}{n} + 1 + \frac{2}{2\sqrt{n}} \operatorname{Re}(w) + \frac{|w|^2}{n} \right) \\ &= n + \sqrt{n}c + \frac{1}{2} (|z|^2 + |w|^2)\end{aligned}\tag{2.4}$$

where  $c = \operatorname{Re}(z + w)$ . We then have that

$$\frac{n\zeta\bar{\eta}}{\lambda} = \frac{(1 + \frac{z}{\sqrt{n}})(1 + \frac{\bar{w}}{\sqrt{n}})}{1 + \frac{1}{\sqrt{n}} \operatorname{Re}(z + w) + \frac{1}{2n}(|z|^2 + |w|^2)}.$$

To expand this in  $n$ , we first recall that  $(1 + \varepsilon)^{-1} = 1 - \varepsilon + \varepsilon^2 + O(\varepsilon^3)$ . Then with the above denominator being expanded we have that  $O(\varepsilon) = O(\frac{1}{\sqrt{n}})$  and so

$$\begin{aligned}\frac{n\zeta\bar{\eta}}{\lambda} &= (1 + \frac{z}{\sqrt{n}})(1 + \frac{\bar{w}}{\sqrt{n}}) \left( 1 - \frac{\operatorname{Re}(z + w)}{\sqrt{n}} - \frac{|z|^2 + |w|^2}{2n} + \frac{(\operatorname{Re}(z + w))^2}{n} \right) + O(n^{-3/2}) \\ &= 1 + \frac{1}{\sqrt{n}}(z + \bar{w} - \operatorname{Re}(z + w)) + \frac{1}{n}a + O(n^{-3/2})\end{aligned}\tag{2.5}$$

where  $a = \frac{-1}{2}(|z|^2 + |w|^2) + (\operatorname{Re}(z + w))^2 + z\bar{w} - (z + \bar{w})\operatorname{Re}(z + w)$ . Writing  $z = z_1 + iz_2$  and  $w = w_1 + iw_2$  with  $z_1, z_2, w_1, w_2 \in \mathbb{R}$ , the coefficient for the  $\frac{1}{\sqrt{n}}$  term can be written as

$$z + \bar{w} - \operatorname{Re}(z + w) = z_1 + iz_2 + w_1 - iw_2 - z_1 - w_1 = i(z_2 - w_2) = i \operatorname{Im}(z + \bar{w}) = ib.$$

Returning to equation (2.3), we first wish to determine the limit of  $A_n$  as  $n \rightarrow \infty$ . This is done by using the asymptotics of (2.4) and (2.5) as

$$A_n = \left( 1 + \frac{ib}{\sqrt{n}} + \frac{a}{n} + O(n^{-3/2}) \right)^{n + \sqrt{n}c + O(1)}.$$

We look at the asymptotic behavior of  $\log A_n$  for simplicity and infer the asymptotic behavior of  $A_n$ . Recall that  $\log(1 + \varepsilon) = \varepsilon - \frac{\varepsilon^2}{2} + O(\varepsilon^3)$  for small  $\varepsilon$ .

$$\begin{aligned}\log A_n &= (n + c\sqrt{n} + O(1)) \log \left( 1 + \frac{ib}{\sqrt{n}} + \frac{a}{n} + O(n^{-3/2}) \right) \\ &= (n + c\sqrt{n} + O(1)) \left( \frac{ib}{\sqrt{n}} + \frac{a}{n} + \frac{b^2}{2n} + O(n^{-3/2}) \right) \\ &= ib\sqrt{n} + ibc + a + \frac{b^2}{2} + O\left(\frac{1}{\sqrt{n}}\right) \\ \implies A_n &= e^{ib\sqrt{n} + b^2/2 + ibc + a + o(1)}.\end{aligned}$$

The term  $ibc + a$  can be simplified as follows

$$\begin{aligned}ibc + a &= i \operatorname{Im}(z + \bar{w}) \operatorname{Re}(z + w) - \frac{1}{2}(|z|^2 + |w|^2) + (\operatorname{Re}(z + w))^2 + z\bar{w} - (z + \bar{w})\operatorname{Re}(z + w) \\ &= z\bar{w} - \frac{1}{2}(|z|^2 + |w|^2) + \operatorname{Re}(z + w) [\operatorname{Re}(z + w) + i \operatorname{Im}(z + \bar{w}) - (z + \bar{w})] \\ &= z\bar{w} - \frac{1}{2}(|z|^2 + |w|^2)\end{aligned}$$

where we in the last step used that  $\operatorname{Re}(z+w) = \operatorname{Re}(z+\bar{w})$ . We can now write  $A_n$  as

$$A_n = e^{ib\sqrt{n}} e^{b^2/2} G(z, w) (1 + o(1))$$

since  $e^{o(1)} = 1 + o(1)$ . The next step is to compute  $B_n$  as  $n \rightarrow \infty$ . Recall that  $Y_n$  converges to the standard normal so the probability density function is of the form  $\frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$ . We will also use that  $\alpha_n = (n-\lambda)/\sqrt{\lambda} \rightarrow -\operatorname{Re}(z+w)$  and that  $\sqrt{\lambda}$  behaves as  $\sqrt{n}$  as  $n \rightarrow \infty$  as is clear from (2.4). Taking  $\sim$  to mean asymptotic equality as  $n \rightarrow \infty$  we have

$$\begin{aligned} \mathbb{E} \left[ \left( \frac{n\zeta\bar{\eta}}{\lambda} \right)^{\sqrt{\lambda}Y_n} \cdot \mathbf{1}_{\{Y_n < \alpha_n\}} \right] &\sim \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha_n} \left( 1 + \frac{i}{\sqrt{n}} \operatorname{Im}(z+\bar{w}) + O\left(\frac{1}{n}\right) \right)^{\sqrt{nt}} e^{-\frac{t^2}{2}} dt \\ &\sim \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\operatorname{Re}(z+\bar{w})} e^{i\operatorname{Im}(z+\bar{w})t} e^{-\frac{t^2}{2}} dt = I. \end{aligned}$$

To compute the last integral we will need to complete the square in the exponent. We have  $ibt - t^2/2 = -b^2/2 - (ib-t)^2/2$ . The substitution  $\zeta = ib - t \implies dt = -d\zeta$  yields

$$I = \frac{-e^{-b^2/2}}{\sqrt{2\pi}} \int_{\infty}^{i\operatorname{Im}(z+\bar{w}) - (-\operatorname{Re}(z+\bar{w}))} e^{-\zeta^2/2} d\zeta = \frac{e^{-b^2}}{\sqrt{2\pi}} \int_{z+\bar{w}}^{\infty} e^{-\zeta^2/2} d\zeta,$$

completing the proof.  $\square$

We conclude that limiting correlation kernel and the associated one-point intensity function are of the form

$$\begin{aligned} K_n(z, w) &= e^{i\operatorname{Im}(z+\bar{w})\sqrt{n}} G(z, w) F(z+\bar{w}) (1 + o(1)) \\ \implies R(z) &= F(2\operatorname{Re}(z)) = \frac{1}{2} \operatorname{erfc}\left(\sqrt{2}\operatorname{Re}(z)\right) \end{aligned} \quad (2.6)$$

which is in agreement with the universal result obtained in [26]. Note that the one-point intensity function is symmetric with respect to translation in the direction of the imaginary axis. This is to be expected from the fact that as zooming in to a point on the boundary of a disk, the edge becomes increasingly straight. We plot the function below.

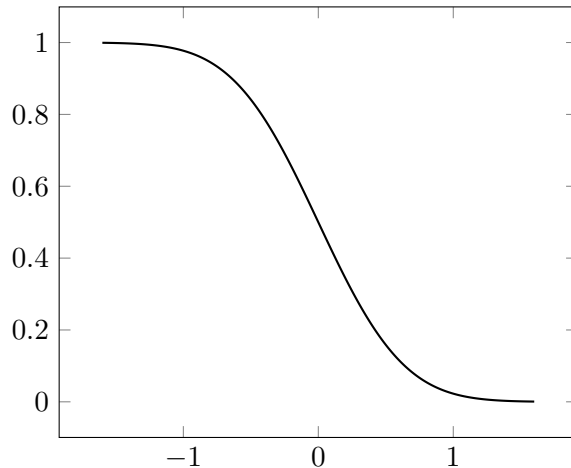


Figure 2.4:  $R(\operatorname{Re}(z))$  at the boundary of the Ginibre ensemble.

From the plot we conclude that the majority of the decay at the edge happens within  $\operatorname{Re}(z) \in (-1, 1)$  which corresponds to a width of approximately  $\frac{2}{\sqrt{n}}$ .

### 2.3.2 Numerical one-point intensity function

We can also calculate (2.2) numerically at the boundary for different  $\lambda$ . Changing  $c$  does not affect the behavior at  $\partial S$  as  $n \rightarrow \infty$  so we only show  $c = 0$  below.

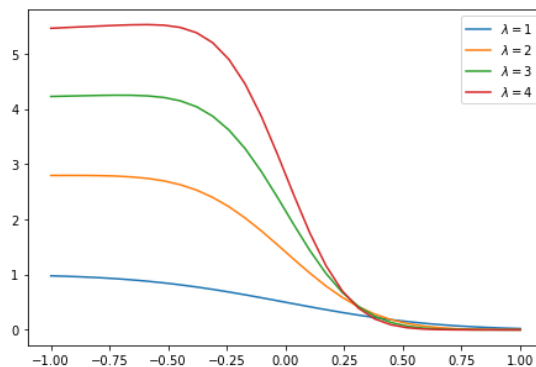


Figure 2.5: Numerical  $R_n(z)$  for a collection of  $\lambda$ 's with  $n = 40\,000$ .

We note that higher  $\lambda$ , higher  $n$  was required in order to make the leftmost part of the plot flat due to the fact that  $\Delta Q$  is more aggressive near the boundary for higher  $\lambda$ . The different scales is a direct result of us discarding the factor  $\frac{1}{\sqrt{2\lambda^{1/\lambda}}}$  in  $r_n$ . Other than that we seem to have recovered the erfc behavior as was to be expected from the universal result in [26].

## Chapter 3

# One-point intensity function of the lemniscate ensemble

Consider the case of the *lemniscate ensemble* which is characterized by

$$Q(\zeta) = |\zeta|^{2k} - 2k^{-1/2} \operatorname{Re}(\zeta^k) = \left| \zeta^k - \frac{1}{\sqrt{k}} \right|^2 - \frac{1}{k}. \quad (3.1)$$

Note that this potential is of the form  $Q(\zeta) = Q_0(\zeta) + h(\zeta)$  where  $h$  is harmonic so this can be seen as a sort of perturbation of the Mittag-Leffler ensemble with the same  $\Delta Q$ . It turns out that the droplet takes the form  $S = \{\zeta \in \mathbb{C}, Q(\zeta) \leq 0\}$  which we will show in Section 3.1. This shape is known as a *lemniscate* and is illustrated for different  $k$  below.

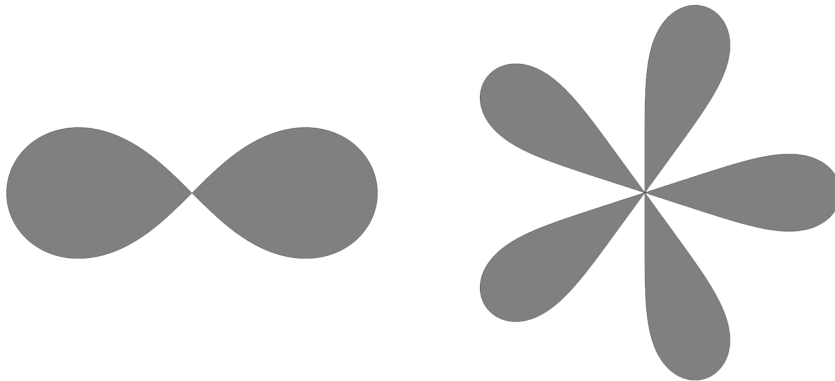


Figure 3.1: The lemniscate ensemble droplet for  $k = 2, 5$ .

We wish to determine the rescaled one-point intensity function  $R$  in a neighborhood of 0. This problem was discussed briefly in [5] while in [11] asymptotic properties of the associated orthogonal polynomials were investigated. While we will not be able to settle the question here, we do obtain partial results and a good approximation which will be interesting to compare to future results.

Recall from (1.16) and (1.15) that the rescaled one-point intensity function and the criteria for the microscopic scale is given by

$$R_n(\zeta) = r_n^2 \sum_{j=0}^{n-1} |p_j(\zeta)|^2 e^{-nQ(\zeta)}, \quad n \int_{\mathbb{D}(p, r_n)} \Delta Q \, dA = 1. \quad (3.2)$$

We calculate  $\Delta Q = k^2|\zeta|^{2k-2}$  and note that  $p = 0$  is a singular point i.e.  $\Delta Q(0) = 0$ . When computing the integral defining  $r_n$  we are not really looking for a specific value but rather determining how  $r_n$  depends on  $n$ . We therefore make the simplification of incorrectly assuming  $\mathbb{D}(0, r_n) \subset S$  since that only changes the result by a factor of 2.

$$1 = n \int_{\mathbb{D}(0, r_n)} k^2 |\zeta|^{2k-2} dA = 2n \int_0^{r_n} r^{2k-1} dr = 2n \frac{r_n^{2k}}{2k} \implies r_n = \left(\frac{k}{n}\right)^{1/2k}.$$

We also discard the factor  $k^{1/2k}$  as it is independent of  $n$ , leaving us with  $r_n = n^{-1/2k}$ .

### Heuristic arguments for the behavior of $R(z)$

Before proceeding and attempting to construct the orthogonal polynomials we can make an educated guess as to how  $R(z)$  should behave. Since the dominating term in the lemniscate potential is  $|\zeta|^{2k}$  which we recall from the Mittag-Leffler potential, we might guess that the one-point intensity function shares similarities with the one in Section 2.2 with  $c = 0$ . From [26] we know that along the straight lines of the lemniscate we should obtain the same erfc behavior as along the Ginibre ensemble edge (see Section 2.3.1). With this in mind we can guess that the  $k = 2$  one-point intensity should be similar to

$$\hat{R}(z) = \underbrace{\frac{1}{2} \operatorname{erfc}\left(\sqrt{2} \operatorname{Re}(z^2)\right)}_{\text{Generalization of eq. (2.6)}} \cdot \underbrace{2E_{\frac{1}{2}, \frac{1}{2}}(|z|^2) e^{-|z|^4}}_{\text{Eq.(2.1)}} \quad (3.3)$$

where the  $\operatorname{Re}(z^2)$  in erfc is only positive in the two sectors  $\{-\frac{\pi}{4} < \arg(z) < \frac{\pi}{4}\}$  and  $\{\frac{3\pi}{4} < \arg(z) < \frac{5\pi}{4}\}$  which is what we want. For larger  $k$ , we could change the erfc argument to target the correct sectors as well as use (2.1) with higher  $k$ .

Since this mainly accounts for the behavior very near 0 and at infinity, we should not expect this to be completely correct but we can still compare it to numerical results for verification and possible clues regarding future modifications.

## 3.1 Determining the droplet

We present some general arguments on a sort of potential with rotational symmetry. This will be partly done in the language of potential theory so we assume that the reader has read Section 1.4 and refer to [34] for more background.

Following [9], we introduce the  $d$ -fold rotated measure  $\mu^{(d)}$  as

$$\mu^{(d)} = \frac{1}{d} \sum_{k=0}^{d-1} \mu_k^{(d)}, \quad \mu_k^{(d)}(B) = \mu(\varphi^{-1}(B \cap S_k))$$

where

$$S_k = \left\{ \zeta \in \mathbb{C}, \frac{2\pi k}{d} \leq \arg \zeta < \frac{2\pi(k+1)}{d} \right\}, \quad k = 0, \dots, d-1$$

is the  $k$ :th sector out of  $d$  sectors of  $\mathbb{C}$  and  $\varphi_k : \mathbb{C} \rightarrow S_k, re^{i\theta} \mapsto r^{\frac{1}{d}} e^{\frac{i\theta}{d}} e^{\frac{2\pi ik}{d}}$ . From this we can see that  $\zeta \in \operatorname{supp} \mu^{(d)} \iff \zeta^d \in \operatorname{supp} \mu$ . Recall from Section 1.4 that the *logarithmic potential*  $U^\mu$  of a measure  $\mu$  is defined as

$$U^\mu(\zeta) = \int \log \frac{1}{|\zeta - \eta|} d\mu(\eta).$$

**Lemma 3.1.1** ([9], Proposition 1). *The  $d$ -fold rotated measure  $\mu^{(d)}$  of the measure  $\mu$  has the logarithmic potential*

$$U^{\mu^{(d)}}(\zeta) = \frac{1}{d}U^\mu(\zeta^d).$$

*Proof.* Substituting the definition of the  $d$ -fold rotated measure directly into the definition of the logarithmic potential we have

$$U^{\mu^{(d)}}(\zeta) = \int_{\mathbb{C}} \log \frac{1}{|\zeta - \eta|} d\mu^{(d)}(\eta) = \frac{1}{d} \sum_{k=0}^{d-1} \int_{\mathbb{C}} \log \frac{1}{|\zeta - \eta|} d\mu_k^{(d)}(\eta).$$

Next we note from the  $B \cap S_k$  in the definition of  $\mu_k^{(d)}$  that we can instead write

$$U^{\mu^{(d)}}(\zeta) = \frac{1}{d} \sum_{k=0}^{d-1} \int_{S_k} \log \frac{1}{|\zeta - \eta|} d\mu_k^{(d)}(\eta)$$

since the  $S_k$ 's are pairwise disjoint. Working backwards from the definition of  $\mu_k^{(d)}$  we see that

$$U^{\mu^{(d)}}(\zeta) = \frac{1}{d} \sum_{k=0}^{d-1} \int_{\mathbb{C}} \log \frac{1}{|\zeta - \varphi_k(\eta)|} d\mu(\eta) = \frac{1}{d} \int_{\mathbb{C}} \sum_{k=0}^{d-1} \log \frac{1}{|\zeta - \varphi_k(\eta)|} d\mu(\eta).$$

In an effort to remove the sum we note that we can exchange it for a product which we move inside the logarithm. The next step is to show that

$$\prod_{k=0}^{d-1} (\zeta - \varphi_k(\eta)) = \zeta^d - \eta.$$

If we consider the above expression as a function of  $\zeta$  with zeros at  $\varphi_k(\eta)$  for some  $\eta$  it is easy to see that the function must be a monic polynomial in  $\zeta$  of degree  $d$ . Since the function  $\zeta^d - \eta$  has zeros exactly at  $\varphi_k(\eta)$  since  $(\varphi_k(\eta))^d = \eta$  for all  $k$ , this must be the polynomial and we are done. Returning to the logarithmic potential we have

$$U^{\mu^{(d)}}(\zeta) = \frac{1}{d} \int_{\mathbb{C}} \log \frac{1}{|\zeta^d - \eta|} d\mu(\eta) = \frac{1}{d}U^\mu(\zeta^d)$$

as was to be shown. □

Armed with the above lemma, we are ready to prove the following proposition.

**Proposition 3.1.2** ([9], Lemma 1). *If the admissible potential  $Q(\zeta)$  can be written in terms of another admissible potential  $V(\zeta)$  as*

$$Q(\zeta) = \frac{1}{d}V(\zeta^d) \tag{3.4}$$

*for some integer  $d$ , then the equilibrium measure for  $Q$  is given by the  $d$ -fold rotated equilibrium measure of  $V$ , i.e.*

$$\mu_Q = \mu_V^{(d)}.$$

*Proof.* Let  $\mu_V$  denote the equilibrium measure of  $V(\zeta)$  and recall that from Section 1.4.1 that it is uniquely characterized by the inequalities

$$\begin{aligned} V(\zeta) + 2U^{\mu_V}(\zeta) &= \gamma \quad \zeta \in \text{supp } \mu_V \text{ q.e.}, \\ V(\zeta) + 2U^{\mu_V}(\zeta) &\geq \gamma \quad \zeta \notin \text{supp } \mu_V \text{ q.e.} \end{aligned}$$

where  $\gamma$  is known as Robin's constant and we recall that q.e. means that the exceptional set is of capacity zero. Exchanging  $\zeta$  for  $\zeta^d$  in the above expressions yields

$$\begin{aligned} V(\zeta^d) + 2U^{\mu_V}(\zeta^d) &= \gamma \quad \zeta^d \in \text{supp } \mu_V \text{ q.e.}, \\ V(\zeta^d) + 2U^{\mu_V}(\zeta^d) &\geq \gamma \quad \zeta^d \notin \text{supp } \mu_V \text{ q.e.} \end{aligned}$$

From the earlier proposition we have that  $U^\mu(\zeta^d) = d \cdot U^{\mu^{(d)}}(\zeta)$ . Combining this with the relation  $V(\zeta^d) = d \cdot Q(\zeta)$  we find

$$\begin{aligned} Q(\zeta) + 2U^{\mu_V^{(d)}}(\zeta) &= \frac{\gamma}{d} \quad \zeta \in \text{supp } \mu_V^{(d)} \text{ q.e.}, \\ Q(\zeta) + 2U^{\mu_V^{(d)}}(\zeta) &\geq \frac{\gamma}{d} \quad \zeta \notin \text{supp } \mu_V^{(d)} \text{ q.e.} \end{aligned}$$

where we used that  $\zeta \in \text{supp } \mu^{(d)} \iff \zeta^d \in \text{supp } \mu$ . The quantity  $\frac{\gamma}{d}$  can now be seen as the Robin's constant for  $\mu_V^{(d)} = \mu_Q$ .  $\square$

Returning to (3.1), we note that we can write

$$Q(\zeta) = \left| \zeta^k - \frac{1}{\sqrt{k}} \right|^2 - \frac{1}{k} = \frac{1}{k} V(\zeta^k), \quad V(\zeta) = k \left| \zeta - \frac{1}{\sqrt{k}} \right|^2 - 1.$$

Our first observation is that the potential  $V$  is that of a transformed Ginibre ensemble. The  $-1$  term can be discarded and the  $\frac{1}{\sqrt{k}}$  term corresponds to having the ensemble be centered at  $\zeta = \frac{1}{\sqrt{k}}$  while the  $k$  factor can be seen as shrinking the area by a factor  $\frac{1}{k}$ . Since the Ginibre droplet has radius 1 this means that the support of  $\mu_V$  is the circle centered at  $\zeta = \frac{1}{\sqrt{k}}$  of radius  $\frac{1}{\sqrt{k}}$  i.e.  $\left\{ \zeta \in \mathbb{C}, \left| \zeta - \frac{1}{\sqrt{k}} \right|^2 \leq \frac{1}{k} \right\}$ .

It is now clear that the lemniscate droplet is the support of the  $k$ -fold rotated measure  $\mu_V^{(d)}$ . Recall that  $\zeta \in \text{supp } \mu^{(d)} \iff \zeta^d \in \text{supp } \mu$ . Thus letting  $S$  denote the lemniscate droplet we have

$$\zeta \in S \iff \zeta^k \in \text{supp } \mu_V \iff \left| \zeta^k - \frac{1}{\sqrt{k}} \right|^2 - \frac{1}{k} \leq 0 \iff Q(\zeta) \leq 0$$

as promised.

*Remark.* Recall from the remark in Example 1.4.2 that not all droplets can be written as  $\{\zeta \in \mathbb{C}, Q(\zeta) \leq A\}$  for some  $A$  so there is no clear shortcut that we could have used to reach the above result.

## 3.2 Implicitly constructing orthogonal polynomials

Recall from Section 1.3.2 that the orthogonal polynomials which we are interested in are orthogonal with respect to the scalar product

$$\langle p_1, p_2 \rangle = \int_{\mathbb{C}} p_1(\zeta) \overline{p_2(\zeta)} e^{-nQ(\zeta)} dA(\zeta).$$



The method which we will detail in this section is equivalent to the Gram-Schmidt process but is written more compactly as a determinant. We will need to use that the inner product of monomials is symmetric. We prove this as a part of the proposition below which will be useful later.

**Proposition 3.2.1.** *If  $i \not\equiv j \pmod{k}$  then  $\langle \zeta^i, \zeta^j \rangle = 0$ , otherwise*

$$\langle \zeta^i, \zeta^j \rangle = \frac{\Gamma\left(\frac{i+1}{k}\right)}{\Gamma\left(\frac{i-j}{k} + 1\right)} \frac{1}{n^{\frac{1+j}{k}} k^{\frac{i-j}{2k} + 1}} {}_1F_1\left(\frac{i+1}{k}, \frac{i-j}{k} + 1, \frac{n}{k}\right), \quad i \equiv j \pmod{k}$$

where  ${}_1F_1$  is the Kummer confluent hypergeometric function which is defined as

$${}_1F_1(a, b, z) = \sum_{m=0}^{\infty} \frac{(a)_m z^m}{(b)_m m!}, \quad (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$$

where  $(a)_n$  is known as the Pochhammer symbol. Furthermore, the inner product is symmetric i.e.

$$\langle p, q \rangle = \langle q, p \rangle, \quad p, q \in \mathcal{W}_n.$$

In particular, the inner product is symmetric for monomials.

*Proof.* We begin by rewriting the potential to make it easier to deal with

$$Q(\zeta) = |\zeta|^{2k} - 2k^{-1/2} \operatorname{Re}(\zeta^k) = |\zeta|^{2k} - \frac{1}{\sqrt{k}}(\zeta^k + \bar{\zeta}^k).$$

The scalar product can then be written as

$$\langle \zeta^i, \zeta^j \rangle = \int_{\mathbb{C}} \zeta^i \bar{\zeta}^j \exp(-n \left[ |\zeta|^{2k} - \frac{1}{\sqrt{k}}(\zeta^k + \bar{\zeta}^k) \right]) dA(\zeta).$$

Replacing the two  $\exp(\zeta^k)$  terms with their respective Taylor series we obtain

$$\begin{aligned} \langle \zeta^i, \zeta^j \rangle &= \sum_{m,l=0}^{\infty} \int_{\mathbb{C}} \frac{\left(\frac{n\zeta^k}{\sqrt{k}}\right)^m \left(\frac{n\bar{\zeta}^k}{\sqrt{k}}\right)^l}{m! l!} \zeta^i \bar{\zeta}^j e^{-n|\zeta|^{2k}} dA(\zeta) \\ &= \sum_{m,l=0}^{\infty} \frac{\left(\frac{n}{\sqrt{k}}\right)^{m+l}}{m! l!} \int_{\mathbb{C}} \zeta^{i+km} \bar{\zeta}^{j+kl} e^{-n|\zeta|^{2k}} dA(\zeta). \end{aligned}$$

Let  $I_{i,j,m,l}$  denote the above integral,  $\alpha = \min(i+km, j+kl)$  and  $\beta = \max(i+km, j+kl) - \alpha$ . Then

$$I_{i,j,m,l} = \int_{\mathbb{C}} |\zeta|^{2\alpha} \tilde{\zeta}^{\beta} e^{-n|\zeta|^{2k}} dA(\zeta)$$

where  $\tilde{\zeta} = \zeta$  if  $i+km > j+kl$  and  $\bar{\zeta}$  if  $i+km < j+kl$ . Note that in the equality case  $\beta = 0$  we do not have to consider  $\tilde{\zeta}$ . Switching to polar coordinates we have  $\zeta = re^{i\theta}$ ,  $dA(\zeta) = \frac{rdrd\theta}{\pi}$ . Thus

$$I_{i,j,m,l} = \frac{1}{\pi} \int_0^{2\pi} e^{\pm i\beta\theta} d\theta \int_0^{\infty} r^{2\alpha+\beta+1} e^{-nr^{2k}} dr$$

where the  $\pm$  depends on if  $i + km$  is larger than  $j + kl$ . The first integral will only be nonzero only if  $\beta = 0$  which corresponds to  $i \equiv j \pmod{k}$ . Hence the first assertion of the proposition follows.

We can relate  $l$  and  $m$  to switch to a single sum as  $l = \frac{i-j}{k} + m$ . We assume that  $i \geq j$  and start  $m$  at  $m = 0$ . The scalar product can then be written as

$$\langle \zeta^i, \zeta^j \rangle = \sum_{m=0}^{\infty} \frac{\left(\frac{n}{\sqrt{k}}\right)^{2m + \frac{i-j}{k}}}{m! \left(m + \frac{i-j}{k}\right)!} I_{i,j,m,m + \frac{i-j}{k}}.$$

Putting  $\beta = 0$  and writing out  $\alpha$  again we can compute the integral with the substitutions  $t = r^{2k}$  and  $s = nt$  as

$$\begin{aligned} I_{i,j,m,m + \frac{i-j}{k}} &= 2 \int_0^{\infty} r^{2i+2km+1} e^{-nr^{2k}} dr = \frac{1}{k} \int_0^{\infty} t^{\frac{i+k(m-1)+1}{k}} e^{-nt} dt \\ &= \frac{1}{kn} \int_0^{\infty} \left(\frac{s}{n}\right)^{\frac{i+k(m-1)+1}{k}} e^{-s} ds = \frac{1}{kn^{\frac{i+km+1}{k}}} \Gamma\left(\frac{i+km+1}{k}\right) \end{aligned}$$

yielding

$$\langle \zeta^i, \zeta^j \rangle = \sum_{m=0}^{\infty} \frac{\left(\frac{n}{\sqrt{k}}\right)^{2m + \frac{i-j}{k}}}{m! \left(m + \frac{i-j}{k}\right)!} \frac{1}{kn^{\frac{i+km+1}{k}}} \Gamma\left(\frac{i+km+1}{k}\right).$$

Writing  $a = \frac{i+1}{k}$ ,  $b = \frac{i-j}{k}$  we have

$$\sum_{m=0}^{\infty} \frac{n^m n^b \Gamma(a+m)}{m! (m+b)! k k^m k^{b/2} n^a} = C \sum_{m=0}^{\infty} \frac{\Gamma(a+m) \left(\frac{n}{k}\right)^m}{\Gamma(m+b+1) m!}$$

where we have collected factors independent of  $m$  in the factor  $C$ . With this we can get our final expression for the inner product as

$$\begin{aligned} C \sum_{m=0}^{\infty} \frac{\Gamma(a+m) \left(\frac{n}{k}\right)^m}{\Gamma(m+b+1) m!} &= C \frac{\Gamma(a)}{\Gamma(b+1)} \sum_{m=0}^{\infty} \frac{(a)_m \left(\frac{n}{k}\right)^m}{(b+1)_m m!} = C \frac{\Gamma(a)}{\Gamma(b+1)} {}_1F_1\left(a, b+1, \frac{n}{k}\right) \\ &= \frac{\Gamma\left(\frac{i+1}{k}\right)}{\Gamma\left(\frac{i-j}{k}+1\right)} \frac{1}{n^{\frac{1+j}{k}} k^{\frac{i-j}{2k}+1}} {}_1F_1\left(\frac{i+1}{k}, \frac{i-j}{k}+1, \frac{n}{k}\right) \end{aligned}$$

which is what we wished to show.

For symmetry of the inner product we begin by noting that it suffices to show symmetry for monic monomials since the inner product for any two polynomials  $p, q$  can be written as

$$\langle p, q \rangle = \left\langle \sum_{i=0}^n a_i \zeta^i, \sum_{j=0}^m b_j \zeta^j \right\rangle = \sum_{i=0}^n \sum_{j=0}^m a_i b_j \langle \zeta^i, \zeta^j \rangle. \quad (3.5)$$

In general we have that  $\langle \zeta^i, \zeta^j \rangle = \overline{\langle \zeta^j, \zeta^i \rangle}$  and since  $\langle \zeta^i, \zeta^j \rangle$  turned out to be real for  $i \geq j$  we must conclude that it is also real for  $j \geq i$  from which the desired conclusion follows.  $\square$

We now return to constructing orthogonal polynomials. Our goal is to derive an explicit expression for each basis polynomial. To this end we define

$$D_j(z) = \begin{vmatrix} \langle z^0, z^0 \rangle & \langle z^0, z^1 \rangle & \cdots & \langle z^0, z^j \rangle \\ \langle z^1, z^0 \rangle & \langle z^1, z^1 \rangle & \cdots & \langle z^1, z^j \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle z^{j-1}, z^0 \rangle & \langle z^{j-1}, z^1 \rangle & \cdots & \langle z^{j-1}, z^j \rangle \\ 1 & z & \cdots & z^j \end{vmatrix} \quad (3.6)$$

and

$$D_j = \begin{vmatrix} \langle z^0, z^0 \rangle & \langle z^0, z^1 \rangle & \cdots & \langle z^0, z^j \rangle \\ \langle z^1, z^0 \rangle & \langle z^1, z^1 \rangle & \cdots & \langle z^1, z^j \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle z^{j-1}, z^0 \rangle & \langle z^{j-1}, z^1 \rangle & \cdots & \langle z^{j-1}, z^j \rangle \\ \langle z^j, z^0 \rangle & \langle z^j, z^1 \rangle & \cdots & \langle z^j, z^j \rangle \end{vmatrix}. \quad (3.7)$$

We now turn to proving the following proposition.

**Proposition 3.2.2.**  $\left\{ \frac{D_j(z)}{\sqrt{D_{j-1}D_j}} \right\}_{j=0}^{n-1}$  constitutes an orthonormal basis for  $\mathcal{W}_n$ .

*Proof.* We begin by claiming that  $\langle z^i, D_j(z) \rangle = 0$  for  $i < j$ . Indeed,  $D_j(z)$  is a polynomial of degree  $j$  as can be seen from expanding the determinant for  $D_j(z)$  as  $D_j(z) = \sum_{m=0}^j c_m z^m$  where  $c_m$  is the determinant of an appropriate minor of the matrix. Now for  $i < j$

$$\begin{aligned} \langle z^i, D_j(z) \rangle &= \left\langle z^i, \sum_{m=0}^j c_m z^m \right\rangle = \sum_{m=0}^j c_m \langle z^i, z^m \rangle \\ &= \begin{vmatrix} \langle z^0, z^0 \rangle & \langle z^0, z^1 \rangle & \cdots & \langle z^0, z^j \rangle \\ \langle z^1, z^0 \rangle & \langle z^1, z^1 \rangle & \cdots & \langle z^1, z^j \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle z^{j-1}, z^0 \rangle & \langle z^{j-1}, z^1 \rangle & \cdots & \langle z^{j-1}, z^j \rangle \\ \langle z^i, z^0 \rangle & \langle z^i, z^1 \rangle & \cdots & \langle z^i, z^j \rangle \end{vmatrix} = 0 \end{aligned} \quad (3.8)$$

where the last equality is guaranteed by the  $i < j$  condition which implies that two of the rows of the determinant must be equal. All that is left now is to appropriately normalize the  $D_j(z)$ 's. When expanding  $\|D_j(z) = \langle D_j(z), D_j(z) \rangle$ , the only term which will make a contribution is the last one. From (3.6) and (3.7) we see that the coefficient in front of  $z^j$  for  $D_j(z)$  will be  $D_{j-1}$ . Writing this out we see that

$$\langle D_j(z), D_j(z) \rangle = D_{j-1} \langle z^j, D_j(z) \rangle = D_{j-1} \sum_{m=0}^j c_m \langle z^j, z^m \rangle = D_{j-1} D_j \quad (3.9)$$

where we in the last step recognized (3.7). The desired result now follows.  $\square$

We summarize the above results below

$$R_n(\zeta) = n^{-1/k} \sum_{j=0}^{n-1} |p_j(\zeta)|^2 e^{-nQ(\zeta)} = n^{-1/k} e^{-n|\zeta|^{2k} + 2nk^{-1/2} \operatorname{Re}(\zeta^k)} \sum_{j=0}^{n-1} \frac{D_j(\zeta)}{D_{j-i} D_j},$$

$$D_j(z) = \begin{vmatrix} \langle z^0, z^0 \rangle & \langle z^0, z^1 \rangle & \cdots & \langle z^0, z^j \rangle \\ \langle z^1, z^0 \rangle & \langle z^1, z^1 \rangle & \cdots & \langle z^1, z^j \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle z^{j-1}, z^0 \rangle & \langle z^{j-1}, z^1 \rangle & \cdots & \langle z^{j-1}, z^j \rangle \\ 1 & z & \cdots & z^j \end{vmatrix},$$

$$D_j = \begin{vmatrix} \langle z^0, z^0 \rangle & \langle z^0, z^1 \rangle & \cdots & \langle z^0, z^j \rangle \\ \langle z^1, z^0 \rangle & \langle z^1, z^1 \rangle & \cdots & \langle z^1, z^j \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle z^{j-1}, z^0 \rangle & \langle z^{j-1}, z^1 \rangle & \cdots & \langle z^{j-1}, z^j \rangle \\ \langle z^j, z^0 \rangle & \langle z^j, z^1 \rangle & \cdots & \langle z^j, z^j \rangle \end{vmatrix},$$

$$\langle \zeta^i, \zeta^j \rangle = \frac{\Gamma\left(\frac{i+1}{k}\right)}{\Gamma\left(\frac{i-j}{k} + 1\right)} \frac{1}{n^{\frac{1+j}{k}} k^{1+\frac{i-j}{2k}}} {}_1F_1\left(\frac{i+1}{k}, \frac{i-j}{k} + 1, \frac{n}{k}\right), \quad i \equiv j \pmod{k}.$$

Unfortunately, attempts to compute  $R$  explicitly from these equations have been unsuccessful. It would seem that the route of solving Riemann-Hilbert problems to determine properties of the orthogonal polynomials considered in [11] among others is the better way to approach this problem.

### 3.3 Numerical computations

In applications, calculating orthogonal polynomials using the method in Section 3.2 turn out to be ineffective due to numerical instabilities. A numerically stable algorithm which we will instead make use of is *Modified Gram-Schmidt* (MGS). We present some pseudocode of the algorithm for completion.

---

**Algorithm 1** Algorithm for orthogonalizing a set of vectors  $v_1, \dots, v_n$  under some scalar product  $\langle \cdot, \cdot \rangle$ .

---

```

procedure MODIFIED GRAM SCHMIDT( $v_1, \dots, v_n$ )
  for  $i = 1$  to  $n$  do
     $r_{ii} = \|v_i\|$ 
     $q_i = v_i / r_{ii}$ 
    for  $j = i + 1$  to  $n$  do
       $r_{ij} = \langle q_i, v_j \rangle$ 
       $v_j = v_j - r_{ij} q_i$ 
    end for
  end for
  return  $q_1, \dots, q_n$ 
end procedure

```

---

Note that the inner product is calculated using (3.5). With this algorithm we can compute  $p_j$  up to  $j \approx 600$  for  $k = 2$  in a reasonable time which can then in turn be used to calculate  $R_{600}$  using (3.2). Note that MGS is inherently single threaded.

### 3.3.1 Results

#### Lemniscate with $k = 2$

Below is the numerical  $R_{600}$  for the lemniscate ensemble in the square of width  $\Delta z = 1$  which corresponds to  $\Delta\zeta = 1 \cdot 600^{-1/4} \approx 0.2$ .

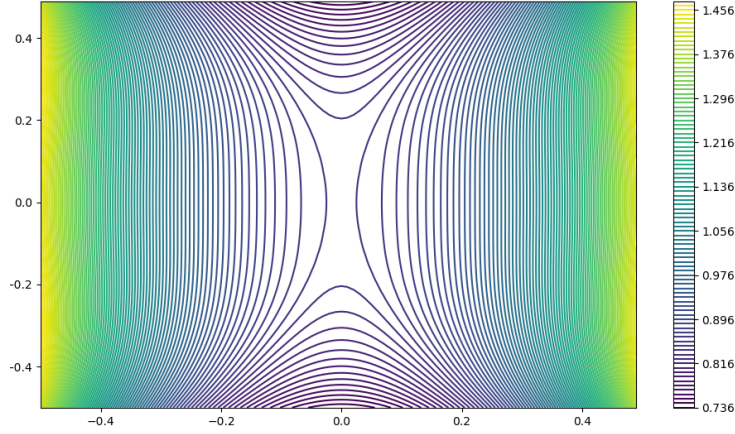


Figure 3.2: Numerical  $R_{600}(z)$  with  $k = 2$ .

We can compare these results to the guess in equation (3.3) by considering the quotient  $\hat{R}(z)/R_n(z)$ . Doing so we obtain the following plot.

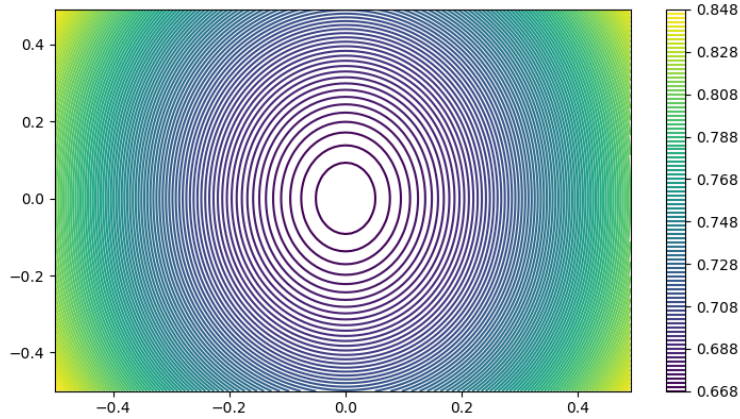


Figure 3.3:  $\hat{R}(z)/R_{600}(z)$  in  $\Delta z = 1$ .

The symmetric appearance is inspiring but it is clear that there is a key ingredient missing. We remark that the resulting quotient is not completely radially symmetric.

We now turn to looking at the point  $\zeta = 0$  in detail. The result  $R_{600}(0) \approx 0.846\ 665$  along with the behavior of  $R_n(0)$  for lower  $n$  suggests the following conjecture.

**Conjecture 3.3.1.** For the  $k = 2$  lemniscate ensemble with the rescaling  $r_n = n^{-1/2k}$  we have

$$R(0) = \frac{3}{2\sqrt{\pi}}.$$

We remark that  $\hat{R}(0) = \frac{1}{\sqrt{\pi}}$  so the fact that there is a factor  $1/\sqrt{\pi}$  in the above equation is not as unexpected as one might initially think.

For higher  $k$ , numerical instabilities related to orthogonalizing the polynomials have made similar results with sufficient precision hard to obtain. The code for orthogonalizing the polynomials along with the coefficients for the  $p_j$ 's for  $k = 2$  can be found in [23].

### Lemniscate with $k = 3$

As mentioned above, high  $n$  leads to numerical instabilities. The highest  $n$  with full orthogonality is  $n = 50$  which suffices for investigating basic characteristics of  $\mathbf{R}_n$ .

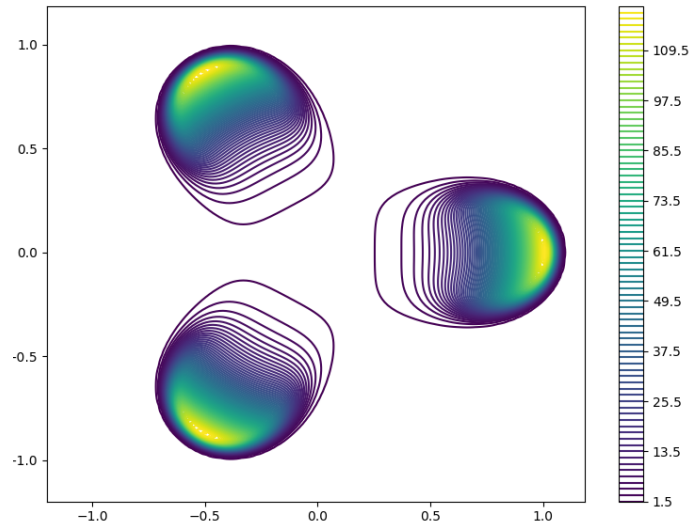


Figure 3.4: Unscaled one-point intensity function  $\mathbf{R}_{50}$  over the entire droplet.

Comparing this with the minimizer in Figure 1.5 we see the same low intensity at the origin. Note that as the amount of particles increases, the peaks at the edges of the droplet become sharper.

We can also look at  $R_{50}$  near  $z = 0$  analogously to Figure 3.3.

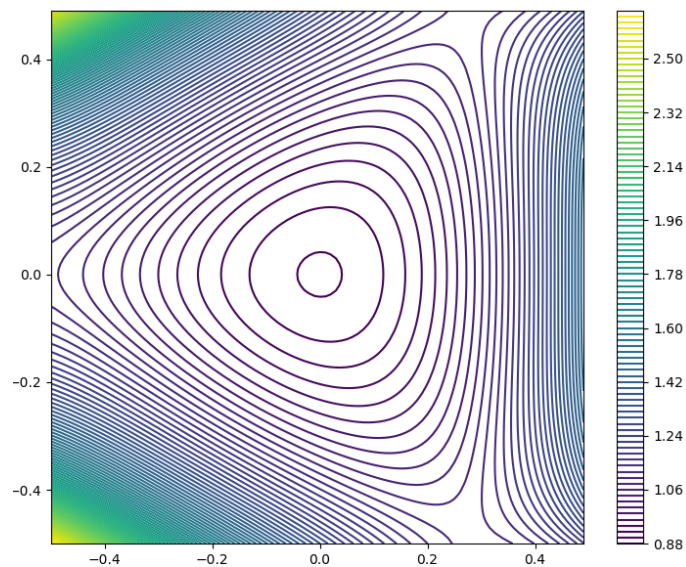


Figure 3.5: Rescaled one-point intensity function  $R_{50}$  near  $z = 0$ .

The  $k = 3$  orthogonal polynomials can also be generated using the code in [23].

### 3.4 Insertion of a point charge at the boundary of the Ginibre ensemble

In this section we consider both the one-point intensity function and the partition function for the case of the Ginibre ensemble with a point charge inserted on the boundary. For the discussion on the partition function we will have to use a result from Chapter 4.

Inserting a point charge on the boundary means that we change the potential to

$$\tilde{Q}(\zeta) = |\zeta|^2 - 2 \operatorname{Re}(\zeta) - \frac{2c}{n} \log |\zeta|.$$

Note that since the Ginibre ensemble is radially symmetric, this can be seen as just inserting a point charge at any point in  $\partial S$ .

#### 3.4.1 Calculating the one-point intensity function

Given a potential  $V$ , we let  $\langle \cdot \rangle_V$  denote the inner product with respect to that potential. We then have that

$$\langle \zeta^i, \zeta^j \rangle_{\tilde{Q}} = \int_{\mathbb{C}} \zeta^i \bar{\zeta}^j e^{-n\tilde{Q}(\zeta)} dA(\zeta) = \int_{\mathbb{C}} \zeta^{i+c} \bar{\zeta}^{j+c} e^{-nQ(\zeta)} dA(\zeta) = \langle \zeta^{i+c}, \zeta^{j+c} \rangle_Q.$$

From this it is clear that for positive integers  $c$ , the one-point intensity function can be approximated using the methods of Section 3.3. Doing this with  $c = 1$  we obtain the following figure where we include the  $c = 0$  case for comparison.

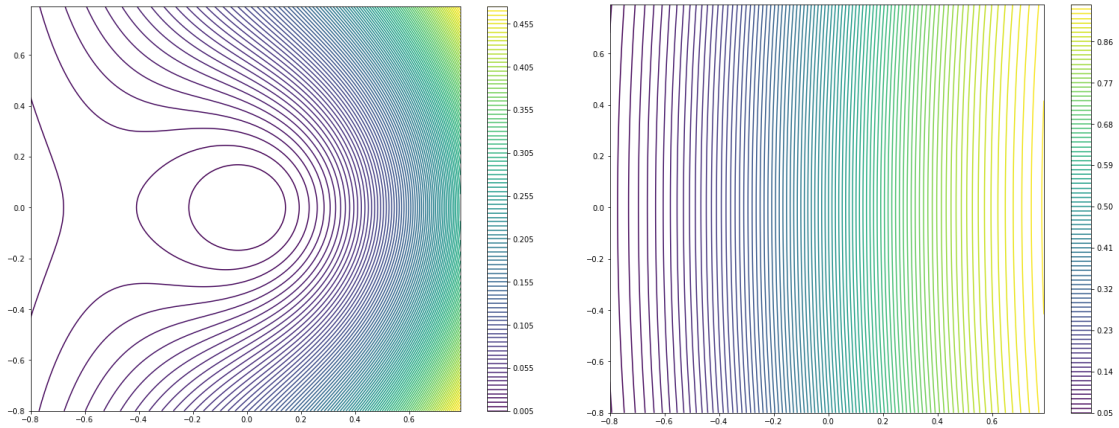


Figure 3.6: Numerical  $R_{150}(z)$  with  $c = 1$ . Figure 3.7: Numerical  $R_{150}(z)$  with  $c = 0$ .

Note that while the two above figures show the same region, their scales differ. Translating in the direction of the imaginary axis in Figure 3.6 sufficiently far, we recover the results of Figure 3.7.

The  $-\frac{2c}{n} \log |\zeta|$  term corresponds to multiplying the correlation kernel by  $|\zeta|^{2c}$  as is clear from considering  $e^{-nQ}$ . Therefore an educated guess for  $R$  in view of the results of Section 2.3.1 could be

$$R_q(z) = \frac{1}{2} \operatorname{erfc} \left( \sqrt{2} \operatorname{Re}(z) \right) \cdot |z|^2. \quad (3.10)$$

Of course this guess diverges as  $z \rightarrow \infty$  on the left half plane so we should not expect it to be completely correct. To check the guess we plot the quantity  $R_g(z)/R_n(z)$ . Doing so we obtain the following figure.

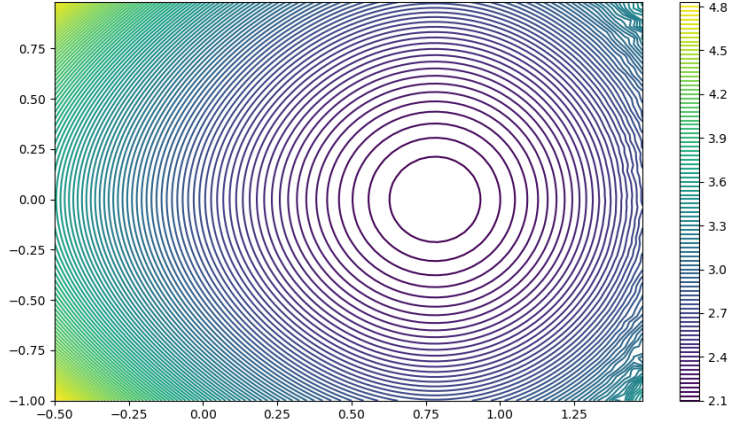


Figure 3.8: Numerical  $R_g(z)/R_{150}(z)$  with  $c = 1$ .

Clearly this quotient is not identically 1 but note that the area around  $z = 0$  is smooth so there is reason to believe that further corrections should reduce to something similar to (3.10) near 0. The singularity near  $z = 0.8$  is not easily identifiable and attempts to compensate for it by another factor have proved unfruitful.

### 3.4.2 Calculating the partition function

We can define the interpolating generalization

$$Q_t(\zeta) = |\zeta - t|^2, \quad \tilde{Q}_t(\zeta) = |\zeta - t|^2 - \frac{2c}{n} \log |\zeta|.$$

This naturally leads to the following  $t$ -dependent quantities.

$$d\tilde{\mathbb{P}}_n^t = \frac{1}{\tilde{Z}_n^t} e^{-\tilde{H}_n^t} dV_n, \quad \tilde{H}_n^t = \sum_{i \neq j} \log \frac{1}{|\zeta_i - \zeta_j|} + n \sum_{j=1}^n \tilde{Q}_t(\zeta_j), \quad \tilde{Z}_n^t = \int_{\mathbb{C}^n} e^{-\tilde{H}_n^t} dV_n.$$

The goal of this section is to determine  $\log \tilde{Z}_n^1$  since this is the partition function of the Ginibre ensemble with a charge at the boundary. We will need to use the result in equation (4.5) from Chapter 4 to determine  $\log \tilde{Z}_n^0$ . To determine  $\log \tilde{Z}_n^t$  we now look at the derivative of  $\log \tilde{Z}_n^t$  with respect to  $t$  as follows

$$\begin{aligned} \frac{d}{dt} \log \tilde{Z}_n^t &= \frac{1}{\tilde{Z}_n^t} \int_{\mathbb{C}^n} \frac{\partial}{\partial t} e^{-\tilde{H}_n^t} dV_n = \int_{\mathbb{C}^n} \frac{\partial}{\partial t} \left( -n \sum_{j=1}^n |\zeta_j - t|^2 \right) d\tilde{\mathbb{P}}_n^t \\ &= \int_{\mathbb{C}^n} 2n \sum_{j=1}^n \operatorname{Re}(\zeta_j - t) d\tilde{\mathbb{P}}_n^t = 2n \operatorname{Re} \int_{\mathbb{C}} (t - \zeta) \tilde{\mathbf{R}}_n^t(\zeta) dA(\zeta) \end{aligned}$$

where  $\tilde{\mathbf{R}}_n^t$  is the non-rescaled one-point intensity function in the potential  $\tilde{Q}_t$ . We will need to make use of the following theorem from [6] to proceed.



**Theorem 3.4.1** ([6], Theorem 6.1). *Let  $\tilde{\mathbf{R}}_n^t$  be as above and let  $\mathbf{R}_n^t$  denote the one-point intensity function associated to the potential  $Q_t$ . Then for  $|t| < 1$*

$$(\tilde{\mathbf{R}}_n^t - \mathbf{R}_n^t) dA = c(\omega_\infty - \delta_0) + o(n)$$

where  $\omega_\infty = \frac{ds}{2\pi}$  is the arclength measure on the circle of radius 1 centered at  $t$  and  $\delta_0$  the Dirac measure at 0.

By symmetry we have that

$$\int_{\mathbb{C}} (t - \zeta) \mathbf{R}_n^t(\zeta) dA(\zeta) = 0$$

since  $\mathbf{R}_n^t$  is radially symmetric around  $t$ . We thus have

$$\begin{aligned} \frac{d}{dt} \log \tilde{Z}_n^t &= 2n \operatorname{Re} \int_{\mathbb{C}} (t - \zeta) (\tilde{\mathbf{R}}_n^t(\zeta) - \mathbf{R}_n^t(\zeta)) dA(\zeta) \\ &= 2cn \operatorname{Re} \left[ \int_{\mathbb{T}_t} (t - \zeta) \frac{ds}{2\pi} - \int_{\mathbb{C}} (t - \zeta) \delta_0(\zeta) dA(\zeta) \right] + o(n) = 0 - 2cnt + o(n) \end{aligned}$$

where  $\mathbb{T}_t$  is the circle of radius 1 centered at  $t$ . With this simple expression we can calculate  $\log \tilde{Z}_n^1$  as

$$\begin{aligned} \log \tilde{Z}_n^1 &= \log \tilde{Z}_n^0 + \int_0^1 (-2cnt) dt + o(n) = \log \tilde{Z}_n^0 - cn + o(n) \implies \\ -\log \tilde{Z}_n^1 &= n^2 \frac{3}{4} - \frac{n}{2} \log n + n \left[ 1 - \frac{\log(2\pi)}{2} + c + c \right] + o(n). \end{aligned}$$

*Remark.* Work is being done on the partition function of less well-behaved ensembles including the lemniscate ensemble by Alfredo Deaño and Nick Simm using Riemann-Hilbert techniques [13].

## Chapter 4

# Partition function of the Mittag-Leffler ensemble

Recall from Chapter 2 that we refer to potentials of the form

$$Q(\zeta) = |\zeta|^{2\lambda} - \frac{2c}{n} \log |\zeta|$$

as *Mittag-Leffler potentials*. The  $\log |\zeta|$  term corresponds to an inserted point charge of strength  $c$  at 0 as can be seen from  $e^{-nQ(\zeta)} = |\zeta - 0|^{2c} e^{-n|\zeta|^{2\lambda}}$ . For technical reasons we will only consider  $c > -1$  but having the charge be repelling is more physically interesting so this will not prove to be an issue.

Our goal of this chapter is to obtain the asymptotic expansion of the partition function of the Mittag-Leffler ensemble. We will also compare the results to those given by Zabrodin and Wiegmann in [38].

The partition function is discussed briefly in Section 1.2.1 but we expand on this by remarking that the logarithm of  $\frac{1}{Z_n}$  can be written as

$$-\log Z_n = An^2 + Bn \log n + Cn + O(\log n).$$

Here  $A$  is known as the free energy of the system and coincides with the logarithmic energy discussed in Section 1.4,  $B$  is universal in that it is independent of  $Q$ . The last named term,  $C$ , can be decomposed as  $C = c_0 + c_1$  where  $c_0$  is independent of  $Q$  and  $c_1$  is proportional to the entropy of the system. See e.g. [25] and [38] for more on this interpretation.

We briefly state that for a measure  $\mu$  on  $\mathbb{C}$  with  $d\mu(\zeta) = m(\zeta) dA(\zeta)$ , the entropy  $\text{ent}(\mu)$  is often defined as

$$\text{ent}(\mu) = \int_{\text{supp}(\mu)} \log \frac{1}{m(\zeta)} d\mu(\zeta) = \int_{\text{supp}(\mu)} m(\zeta) \log \frac{1}{m(\zeta)} dA(\zeta) \quad (4.1)$$

which can be seen as a continuous counterpart to (1.3). In particular, the Ginibre ensemble, which is the case  $c = 0, \lambda = 1$ , has entropy 0 since  $\log \frac{1}{\Delta Q} \equiv 0$  which will allow us to determine  $c_0$  and by extension  $c_1$  for other systems. The entropy interpretation is however slightly problematic as will be discussed in Section 4.2.

### 4.1 Explicit calculation of the asymptotic expansion

Our main tool to calculate the partition function is (1.10). Since we calculated the squares of the orthogonal polynomials for the Mittag-Leffler ensemble in Lemma 2.1.1

we can proceed directly and write

$$Z_n = \frac{\left(\frac{c+1-\lambda}{\lambda}\right)! \cdot \left(\frac{c+1-\lambda}{\lambda} + \frac{1}{\lambda}\right)! \cdots \left(\frac{c+1-\lambda}{\lambda} + \frac{n-1}{\lambda}\right)!}{(\lambda n^{\frac{c}{\lambda}})^n n^{\frac{0+1+2+\dots+n}{\lambda}}} n!. \quad (4.2)$$

To unravel this messy expression we consider the logarithm of  $Z_n$  which turns the product in the numerator into a sum

$$\log \left[ \prod_{k=0}^{n-1} \left(\frac{c+1-\lambda+k}{\lambda}\right)! \right] = \sum_{k=0}^{n-1} \log \left[ \left(\frac{c+1-\lambda+k}{\lambda}\right)! \right]. \quad (4.3)$$

Throughout this chapter we will make frequent use of the asymptotic expansion of the factorial function. This expansion is often referred to as *Stirling's approximation* and takes on many equivalent forms, one of them being

$$\log n! = n \log n - n + \frac{1}{2} \log(2\pi) + O\left(\frac{1}{n}\right)$$

as found in e.g. [33, Eq. 5.11.1]. Applying Stirling's approximation to (4.3) we have that each term takes the form

$$\begin{aligned} \log \left[ \left(\frac{c+1-\lambda+k}{\lambda}\right)! \right] &= \left(\frac{c+1-\lambda+k}{\lambda}\right) \log \left(\frac{c+1-\lambda+k}{\lambda e}\right) \\ &+ \frac{1}{2} \left[ \log(2\pi) + \log \left(\frac{c+1-\lambda+k}{\lambda}\right) \right] + O\left(\frac{\lambda}{c+1-\lambda+k}\right). \end{aligned}$$

Stirling's approximation requires the argument to be large but this is not an issue as we only need to consider the tail of the sum for which we can safely assume  $k \gg \lambda$ . The ordo term above is of the order of  $\frac{1}{k}$  which grows as  $\log n$  when summed and so a  $O(\log n)$  term in the expansion of  $\log Z_n$  will be able to account for its behavior. Returning to the expansion we move the logarithms outside again and write

$$\underbrace{\log \left[ \prod_{k=0}^{n-1} \left(\frac{c+1-\lambda+k}{\lambda e}\right)^{\left(\frac{c+1-\lambda+k}{\lambda}\right)} \right]}_{=A_1} + \frac{1}{2} \underbrace{\log \left[ \prod_{k=0}^{n-1} \left(\frac{c+1-\lambda+k}{\lambda}\right) \right]}_{=\frac{1}{2}A_2} + \underbrace{n \frac{\log(2\pi)}{2}}_{=A_3} + O(\log n).$$

We denote the terms  $A_1, A_2$  and  $A_3$  to make it easier to keep track of them and begin by expanding  $A_1$ .

**Lemma 4.1.1.**  $A_1$  can be simplified as

$$\begin{aligned} \log \left[ \prod_{k=0}^{n-1} \left(\frac{c+1-\lambda}{\lambda e} + \frac{k}{\lambda e}\right)^{\left(\frac{c+1-\lambda+k}{\lambda}\right)} \right] \\ = \frac{1}{\lambda} \left[ \zeta'(-1, c+1-\lambda+n) - \zeta'(-1, c+1-\lambda) \right] - \frac{n}{2\lambda} (1+2c-2\lambda+n)(1+\log \lambda) \end{aligned}$$

where  $\zeta'$  is the derivative with respect to the first parameter of the generalized Riemann zeta function which is defined as

$$\zeta(s, q) = \sum_{k=0}^{\infty} (q+k)^{-s}.$$

*Proof.* Expanding  $A_1$  back into a sum we have

$$\begin{aligned}
A_1 &= \sum_{k=0}^{n-1} \left( \frac{c+1-\lambda+k}{\lambda} \right) [\log(c+1-\lambda+k) - \log(\lambda e)] \\
&= \frac{1}{\lambda} \left[ \sum_{k=0}^{n-1} (c+1-\lambda+k) \log(c+1-\lambda+k) - \sum_{k=0}^{n-1} (c+1-\lambda+k) [\log \lambda + 1] \right] \\
&= \frac{1}{\lambda} \sum_{k=0}^{n-1} (q+k) \log(q+k) - \frac{n}{2\lambda} (1+2c-2\lambda+n)(1+\log \lambda)
\end{aligned}$$

where we have defined  $q = c + 1 - \lambda$ . To get the last sum into something which we can expand asymptotically, we begin by noting that with  $\zeta$  the generalized Riemann zeta function we have that

$$\left. \frac{d}{ds} \zeta(s, q) \right|_{s=-1} = \left. \frac{d}{ds} \sum_{k=0}^{\infty} (q+k)^{-s} \right|_{s=-1} = \sum_{k=0}^{\infty} -(q+k) \log(q+k).$$

Taking  $(\cdot)$  to mean the derivative with respect to the first parameter we have that

$$\sum_{k=0}^{n-1} (q+k) \log(q+k) = \underbrace{\sum_{k=0}^{\infty} (q+k) \log(q+k)}_{-\zeta'(-1, q)} - \underbrace{\sum_{k=0}^{\infty} (q+k+n) \log(q+k+n)}_{-\zeta'(-1, q+n)}.$$

Bringing this together we can write

$$A_1 = \frac{1}{\lambda} [\zeta'(-1, c+1-\lambda+n) - \zeta'(-1, c+1-\lambda)] - \frac{n}{2\lambda} (1+2c-2\lambda+n)(1+\log \lambda)$$

which is what we wished to show.  $\square$

The asymptotic behavior of  $\zeta'(-1, x)$  can be found in [18, p. 349] as

$$\zeta'(-1, q) = \frac{1}{2} q^2 \log q - \frac{1}{4} q^2 - \frac{1}{2} q \log q + O(\log q). \quad (4.4)$$

Combining this with the above lemma we can expand  $A_1$  as

$$\begin{aligned}
\frac{1}{\lambda} \zeta'(-1, c+1-\lambda+n) &= \frac{1}{\lambda} \left[ \frac{1}{2} (c+1-\lambda+n)^2 \log(c+1-\lambda+n) - \frac{1}{4} (c+1-\lambda+n)^2 \right. \\
&\quad \left. - \frac{1}{2} (c+1-\lambda+n) \log(c+1-\lambda+n) + O(\log(c+1-\lambda+n)) \right] \\
&= n^2 \log n \frac{1}{2\lambda} - n^2 \frac{1}{4\lambda} + n \log n \frac{2c+1-2\lambda}{2\lambda} + O(\log n), \\
A_1 &= \frac{n^2 \log n}{2\lambda} - n^2 \left[ \frac{3}{4\lambda} + \frac{\log \lambda}{2\lambda} \right] + n \log n \frac{2c+1-2\lambda}{2\lambda} \\
&\quad - n \frac{1}{2\lambda} (1+2c-2\lambda)(1+\log \lambda) + O(\log n).
\end{aligned}$$

The next lemma will allow us to rewrite  $A_2$  to make is easier to expand.

**Lemma 4.1.2.**

$$\prod_{k=0}^{n-1} \left( \frac{c+1-\lambda}{\lambda} + \frac{k}{\lambda} \right) = \frac{\Gamma(c+1-\lambda+n)}{\lambda^n \Gamma(c+1-\lambda)}.$$

*Proof.* After pulling out the factor  $\lambda^{-n}$  we are left with the product  $(c+1-\lambda) \cdots (c+n-\lambda)$  which we can write as

$$\frac{(c+n-\lambda)!}{(c-\lambda)!} = \frac{\Gamma(c+1-\lambda+n)}{\Gamma(c+1-\lambda)}$$

and the result follows after multiplying  $\lambda^{-n}$  back in.  $\square$

With this we can use Stirling's approximation to obtain the asymptotics of  $A_2$ .

$$\begin{aligned} \log \left[ \prod_{k=0}^{n-1} \left( \frac{c+1-\lambda}{\lambda} + \frac{k}{\lambda} \right) \right] &= \log((c-\lambda+n)!) - n \log \lambda - \log((c-\lambda)!) \\ &= (c-\lambda+n) [\log(c-\lambda+n) - 1] - n \log \lambda + O(\log n) \\ &= n(\log n - 1) - n \log \lambda + O(\log n). \end{aligned}$$

Lastly we note that the logarithm of the denominator of (4.2) can be written as

$$n \log \lambda + \frac{cn}{\lambda} \log n + \frac{n(n+1)}{2\lambda} \log n = n^2 \frac{1}{2\lambda} \log n + n \log n \left[ \frac{c}{\lambda} + \frac{1}{2\lambda} \right] + n \log \lambda.$$

We are now ready to expand (4.2). For the  $n!$  factor we use Stirling's approximation again.

$$\begin{aligned} -\log Z_n &= \underbrace{\frac{-n^2 \log n}{2\lambda} + n^2 \left[ \frac{3}{4\lambda} + \frac{\log \lambda}{2\lambda} \right] - n \log n \frac{2c+1-2\lambda}{2\lambda} + \frac{n}{2\lambda} (1+2c-2\lambda)(1+\log \lambda)}_{A_1} \\ &\quad - \underbrace{\frac{n}{2} [\log n - 1] + \frac{n}{2} \log \lambda}_{\frac{1}{2}A_2} + \underbrace{n^2 \frac{1}{2\lambda} \log n + n \log n \left[ \frac{c}{\lambda} + \frac{1}{2\lambda} \right] + n \log \lambda}_{\text{Denominator}} \\ &\quad - \underbrace{\frac{n \log(2\pi)}{2}}_{A_3} - \underbrace{n \log n + n}_{\text{Stirling } n!} + O(\log n) \\ &= n^2 \left[ \frac{3}{4\lambda} + \frac{\log \lambda}{2\lambda} \right] - \frac{n}{2} \log n \\ &\quad + n \left[ 1 - \frac{\log(2\pi)}{2} + \frac{1+2c-\lambda + (1+2c+\lambda) \log \lambda}{2\lambda} \right] + O(\log n). \end{aligned} \tag{4.5}$$

Recalling that the Ginibre ensemble ( $c=0, \lambda=1$ ) has entropy 0 we note that  $1 - \frac{\log(2\pi)}{2}$  is the constant term and so the entropy is  $\frac{1+2c-\lambda + (1+2c+\lambda) \log \lambda}{2\lambda}$ . The  $n \log n$  term is constant as expected and the free energy is of the form  $\frac{3}{4\lambda} + \frac{\log \lambda}{2\lambda}$ .

*Remark.* We can also consider letting  $c$  scale with  $n$ . This corresponds to changing  $c \mapsto cn$  in (4.5) which we write as

$$\begin{aligned} -\log Z_n &= n^2 \left[ \frac{3}{4\lambda} + \frac{\log \lambda}{2\lambda} + \frac{c}{\lambda} (1+\log \lambda) \right] - \frac{n}{2} \log n \\ &\quad + n \left[ 1 - \frac{\log(2\pi)}{2} + \frac{1-\lambda + (1+\lambda) \log \lambda}{2\lambda} \right] + O(\log n). \end{aligned} \tag{4.6}$$

## Elliptic Ginibre ensemble

In this chapter we have limited ourselves to the case of radially symmetric potentials on account of the simpler orthogonal polynomials. However, using some older results of orthogonal polynomials by Eijndhoven and Meyers [36], Lee and Riser [29] have recently treated the case of the potential

$$Q(\zeta) = |\zeta| - t \operatorname{Re}(\zeta^2), \quad 0 \leq t < 1$$

for which the droplet is an ellipse. We refer to this ensemble as the *elliptic Ginibre ensemble*. From equation (15) in [29] we see that

$$\|\pi_j\|^2 = \frac{j!}{\sqrt{n} (n(1-t^2))^{j+1/2}}$$

after discarding a factor  $\pi$  to compensate for our reduced area element. To calculate the partition function we make use of (1.10) and find

$$Z_n = \frac{1}{(\sqrt{n})^n (n - nt^2)^{n/2}} \underbrace{\left( \prod_{j=0}^{n-1} j! \right)}_{=G(n+2)} \cdot \underbrace{\left( \prod_{j=0}^{n-1} \frac{1}{(n - nt^2)^j} \right)}_{=\frac{1}{(n-nt^2)^{\frac{n}{2}(n-1)}}} \implies$$

$$\log Z_n = -\frac{n}{2} \log n - \frac{n}{2} (\log(1-t^2) + \log n) + \log G(n+2) - \frac{n}{2} (n-1) (\log(1-t^2) + \log n)$$

where  $G$  is known as the *Barnes G-function*. The expansion of the logarithm of the Barnes G-function can be found in [10, p. 269] as

$$\log G(z+1) = z^2 \left( \frac{1}{2} \log z - \frac{3}{4} \right) + \frac{1}{2} \log(2\pi)z + O(\log z).$$

Putting this together we obtain

$$-\log Z_n = n^2 \left[ \frac{3}{4} - \frac{t^2}{2} \right] - \frac{n}{2} \log n + n \left[ 1 - \frac{\log(2\pi)}{2} \right] + O(\log n) \quad (4.7)$$

and we conclude that changing the parameter  $t$  affects only the  $n^2$  term. We also note that  $\Delta Q \equiv 1$ , which is the same as for the Ginibre potential, since the real part of an analytic function is harmonic.

## 4.2 Zabrodin and Wiegmann comparison

The asymptotic expansion of  $\log Z_n$  is discussed in a more general setting in [38]. There  $\hbar$  takes on the role of  $1/n$  and the expansion is written as

$$\log Z_n = c(n) + \frac{F_0}{\hbar^2} + \frac{F_{1/2}}{\hbar} + F_1 + O(\hbar) \quad (4.8)$$

where  $c(n)$  is a normalizing term which is independent of  $Q$ . An expression for  $F_{1/2}$  is given in equation (3.6) in [38] as

$$F_{1/2} = -\frac{1}{2}(2-\beta) \int \rho_0 \log \rho_0 d^2 z \quad (4.9)$$

where  $\rho_0$  is the density of particles. In this section we attempt to verify the above relation for the results in the previous section. We first note that (4.8) does not contain a factor  $n \log n$  meaning it has been absorbed by the normalizing term which agrees with (4.5) and (4.7).

The density of particles,  $\rho_0$ , can be identified with  $\Delta Q$  in view of the discussion in Section 1.4. We therefore need to evaluate the following integral

$$I = \int_S \Delta Q \log(\Delta Q) dA$$

which shares a striking similarity to (4.1), lending credibility to the entropy interpretation.

Allowing  $c \neq 0$  for the Mittag-Leffler ensemble introduces issues since we are multiplying  $\Delta Q$  with  $\log \Delta Q$ . We therefore fix  $c = 0$  and recall from Section 2.1.1 that  $\Delta Q = \lambda^2 |\zeta|^{2\lambda-2}$  and  $S = \{\zeta \in \mathbb{C}, |\zeta| \leq \lambda^{\frac{1}{2\lambda}}\}$ . Substituting this we have

$$I = \lambda^2 \left[ (2 \log \lambda) \int_S |\zeta|^{2\lambda-2} dA(\zeta) + (2\lambda - 2) \int_S |\zeta|^{2\lambda-2} \log |\zeta| dA(\zeta) \right].$$

The first integral is easily evaluated as  $1/\lambda^2$ . For the second integral we use integration by parts

$$\begin{aligned} \int_S |\zeta|^{2\lambda-2} \log |\zeta| dA(\zeta) &= 2 \log(r) \frac{r^{2\lambda}}{2\lambda} \Big|_{r=\lambda^{\frac{1}{2\lambda}}} - 2 \int_0^{\lambda^{\frac{1}{2\lambda}}} \frac{r^{2\lambda-1}}{2\lambda} dA(\zeta) \\ &= \frac{-1}{2\lambda^3} \log \lambda - \frac{1}{2\lambda^3} = \frac{-1}{2\lambda^3} (1 + \log \lambda). \end{aligned}$$

Returning we find

$$I = 2 \log \lambda - \frac{\lambda - 1}{\lambda} (1 + \log \lambda).$$

We substitute this into (4.9) with  $\beta = 1$  and obtain

$$F_{1/2} = \frac{\lambda - 1}{2\lambda} (1 + \log \lambda) - \log \lambda = -\frac{1 - \lambda + (1 + \lambda) \log \lambda}{2\lambda}.$$

Comparing with (4.5), we see that we have the same result with a different sign which is a consequence of Zabrodin and Wiegmann expanding  $\log Z_n$  and not  $-\log Z_n$ . We conclude that (4.9) is correct at least in the  $c = 0$  case for Mittag-Leffler ensembles as well as for the elliptic Ginibre ensemble.

We also note that  $F_{1/2}$  is the same for the Ginibre ensemble and the elliptic Ginibre ensemble which is to be expected from the fact that  $\log 1 = 0$ .

### 4.3 Hard edge correction

We refer to ensembles with  $Q = +\infty$  in the complement of  $S$  as *hard edge ensembles*. In this context, ensembles with regular edges are referred to as *free edge ensembles*. For more on hard edge ensembles see e.g. [2]. Our goal of this section is to calculate the partition function of the hard edge Mittag-Leffler ensemble. First we show that the orthogonal polynomials are the monomials analogously to the proof of Lemma 2.1.1. Recall from Section 2.1.1 that  $S = \{\zeta \in \mathbb{C}, |\zeta| \leq \lambda^{\frac{1}{2\lambda}}\}$ .

**Lemma 4.3.1.** *For*

$$Q(\zeta) = \begin{cases} |\zeta|^{2\lambda} - \frac{2c}{n} \log |\zeta| & \text{if } |\zeta| \leq \lambda^{\frac{-1}{2\lambda}}, \\ +\infty & \text{if } |\zeta| > \lambda^{\frac{-1}{2\lambda}}, \end{cases}$$

*the monic orthogonal polynomials are monomials with*

$$\|\pi_j\|^2 = \frac{\gamma\left(\frac{c+j+1}{\lambda}, \frac{n}{\lambda}\right)}{\lambda n^{\frac{c+j+1}{\lambda}}}, \quad \gamma(x, b) = \int_0^b t^{x-1} e^{-t} dt$$

*where  $\gamma$  is known as the lower incomplete gamma function.*

*Proof.* Proceeding as in Lemma 2.1.1, we write the integral as

$$\|\pi_j\|^2 = 2 \int_0^{R_\lambda} r^{2c+2j+1} e^{-nr^{2\lambda}} dr.$$

After making the substitution  $t = nr^{2\lambda}$ , the new limits of integration are 0 and  $\frac{n}{\lambda}$  and the integral takes the form

$$\|\pi_j\|^2 = \frac{1}{\lambda n} \left(\frac{1}{n}\right)^{\frac{c+j+1-\lambda}{\lambda}} \int_0^{\frac{n}{\lambda}} t^{\frac{c+j+1-\lambda}{\lambda}} e^{-t} dt = \frac{\gamma\left(\frac{c+j+1}{\lambda}, \frac{n}{\lambda}\right)}{\lambda n^{\frac{c+j+1}{\lambda}}}.$$

□

Substituting this into (1.10) we find

$$Z_n^{\text{hard}} = \frac{\gamma\left(\frac{c+1}{\lambda}, \frac{n}{\lambda}\right) \cdot \gamma\left(\frac{c+2}{\lambda}, \frac{n}{\lambda}\right) \cdots \gamma\left(\frac{c+n}{\lambda}, \frac{n}{\lambda}\right)}{(\lambda n^{\frac{c}{\lambda}})^n n^{\frac{0+1+2+\dots+n}{\lambda}}} n!.$$

Instead of expanding this directly as in Section 4.1, we consider the quotient

$$q_n = \frac{Z_n^{\text{hard}}}{Z_n^{\text{free}}} = \frac{\gamma\left(\frac{c+1}{\lambda}, \frac{n}{\lambda}\right) \cdot \gamma\left(\frac{c+2}{\lambda}, \frac{n}{\lambda}\right) \cdots \gamma\left(\frac{c+n}{\lambda}, \frac{n}{\lambda}\right)}{\Gamma\left(\frac{c+1}{\lambda}\right) \cdot \Gamma\left(\frac{c+2}{\lambda}\right) \cdots \Gamma\left(\frac{c+n}{\lambda}\right)} = \prod_{j=1}^n q_{j,n}, \quad q_{j,n} = \frac{\gamma\left(\frac{c+j}{\lambda}, \frac{n}{\lambda}\right)}{\Gamma\left(\frac{c+j}{\lambda}\right)}. \quad (4.10)$$

Computing the above product numerically for different  $\lambda$  suggests that  $\frac{\log(q_n)}{\sqrt{n}}$  should converge to  $-\kappa\sqrt{\lambda}$  for some positive constant  $\kappa$  as  $n \rightarrow \infty$ . Our goal is now to give a nicer expression for  $\kappa$  than an infinite product and prove the  $\sqrt{\lambda}$  and  $\sqrt{n}$  dependence.

**Proposition 4.3.2.** *In the asymptotic limit  $n \rightarrow \infty$  we have the relation*

$$q_n = \prod_{j=\sqrt{n} \log n}^n q_{j,n} (1 + o(1)).$$

Before attempting a proof of this fact we present a helpful lemma.

**Lemma 4.3.3.** *The sequence  $\{q_{j,n}\}_{j=1}^n$  is decreasing for  $j + c < n$ .*



*Proof.* For simplicity we introduce the *upper incomplete gamma function* which is defined as

$$\Gamma(a, x) = \int_x^\infty t^{a-1} e^{-t} dt = \Gamma(a) - \gamma(a, x)$$

allowing us to write  $q_{j,n}$  as

$$q_{j,n} = 1 - \frac{\Gamma\left(\frac{c+j}{\lambda}, \frac{n}{\lambda}\right)}{\Gamma\left(\frac{c+j}{\lambda}\right)} =: 1 - Q\left(\frac{c+j}{\lambda}, \frac{n}{\lambda}\right). \quad (4.11)$$

*Remark.* The function  $Q$  is sometimes referred to as the *regularized incomplete gamma function* since  $0 \leq Q \leq 1$ .

It is now clear that if we can prove that  $Q(a, b)$  is increasing in  $a$  with  $a < b$ , the result will follow. Taking  $(')$  to mean the derivative with respect to the first parameter we as usual have

$$Q'(a, b) = \frac{\Gamma'(a, b)\Gamma(a) - \Gamma(a, b)\Gamma'(a)}{(\Gamma(a))^2}.$$

The derivative of the gamma function can be given in terms of the *digamma function*  $\psi$  which is defined as the logarithmic derivative of the gamma function, i.e.

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

From [33, Eq. 5.11.2] we have the inequality

$$\psi(x) < \log x \quad (4.12)$$

with asymptotic equality. We can now calculate  $\Gamma'(a, b)$  as

$$\begin{aligned} \Gamma'(a, b) &= \int_b^\infty \frac{\partial}{\partial a} t^{a-1} e^{-t} dt = \int_b^\infty \log(t) \cdot t^{a-1} e^{-t} dt = - \int_b^\infty \log(t) \frac{\partial \Gamma(a, b)}{\partial b} dt \\ &= - \left[ \Gamma(a, t) \log t \right]_b^\infty + \int_b^\infty \frac{\Gamma(a, t)}{t} dt = \Gamma(a, b) \log b + \int_b^\infty \frac{\Gamma(a, t)}{t} dt \end{aligned}$$

where the disappearance of  $\Gamma(a, t) \log t$  at  $t = \infty$  is justified by the asymptotic identity

$$\Gamma(a, x) = x^{a-1} e^{-x} \left( 1 + O\left(\frac{a}{x}\right) \right) \quad (4.13)$$

as can be found in e.g. [33, Eq. 8.11.2].

Returning to  $Q$ , we take  $x \sim y$  to mean that  $x$  and  $y$  have the same sign. We now claim that  $Q'(a, b) > 0$  if  $a < b$ . Indeed,

$$\begin{aligned} Q'(a, b) &= \frac{\Gamma'(a, b)\Gamma(a) - \Gamma(a, b)\Gamma'(a)}{(\Gamma(a))^2} \sim \Gamma(a, b) \log b + \int_b^\infty \frac{\Gamma(a, t)}{t} dt - \Gamma(a, b)\psi(a) \\ &\sim \log b + \frac{1}{\Gamma(a, b)} \int_b^\infty \frac{\Gamma(a, t)}{t} dt - \psi(a) \\ &> \log b - \log a + \frac{1}{\Gamma(a, b)} \int_b^\infty \frac{\Gamma(a, t)}{t} dt > 0 \end{aligned}$$

where we in the second to last step made use of (4.12). □

We are now ready to prove the earlier proposition.

*Proof of Proposition 4.3.2.* Throughout the proof we will treat  $\sqrt{n} \log n$  as an integer whenever necessary for simplicity. Since the statement of the proposition is that the first  $\sqrt{n} \log n$  factors are dismissable we can restate it as

$$\prod_{j=1}^{\sqrt{n} \log n} q_{j,n} = 1 + o(1) \iff \sum_{j=1}^{\sqrt{n} \log n} \log(q_{j,n}) = o(1).$$

Since  $q_{j,n} \leq 1$ , the largest term in the above sum is the one for which  $q_{j,n}$  is smallest. We can now put an upper bound on the sum by replacing  $q_{j,n}$  with  $q_{\sqrt{n} \log n, n}$  on account of the above lemma.

$$\left| \sum_{j=1}^{\sqrt{n} \log n} \log(q_{j,n}) \right| \leq \left| \sum_{j=1}^{\sqrt{n} \log n} \log(q_{\sqrt{n} \log n, n}) \right| = \sqrt{n} \log n \left| \log(q_{\sqrt{n} \log n, n}) \right|.$$

From now on we drop the absolute value bars since we wish to prove that the above quantity converges to 0 as  $n \rightarrow \infty$ . We now consider the Taylor expansion of  $\log(q_{\sqrt{n} \log n, n})$  which in view of (4.11) takes the form

$$\log(q_{\sqrt{n} \log n, n}) = -\frac{\Gamma\left(\frac{c+j}{\lambda}, \frac{n}{\lambda}\right)}{\Gamma\left(\frac{c+j}{\lambda}\right)} + O\left[\left(\frac{\Gamma\left(\frac{c+j}{\lambda}, \frac{n}{\lambda}\right)}{\Gamma\left(\frac{c+j}{\lambda}\right)}\right)^2\right].$$

If we can show that the first term multiplied by  $\sqrt{n} \log n$  goes to zero as  $n \rightarrow \infty$  it will follow that the ordo term multiplied by  $\sqrt{n} \log n$  also dies as  $n \rightarrow \infty$  since  $\sqrt{n} \log n$  clearly diverges. Before computing this limit we set  $c = 0$  without loss of generality since  $j$  grows as  $\sqrt{n} \log n$  and the additional term  $c$  becomes negligible in the limit. The denominator can be expanded using Stirling's approximation and for the numerator we use (4.13). Dropping the minus sign and expanding the quotient we obtain

$$\begin{aligned} \sqrt{n} \log n \frac{\Gamma\left(\frac{\sqrt{n} \log n}{\lambda}, \frac{n}{\lambda}\right)}{\Gamma\left(\frac{\sqrt{n} \log n}{\lambda}\right)} &= \frac{(\sqrt{n} \log n) \cdot \left(\frac{n}{\lambda}\right)^{\frac{\sqrt{n} \log n}{\lambda} - 1} e^{-\frac{n}{\lambda}} \left(1 + O\left(\frac{\sqrt{n} \log n}{n}\right)\right)}{\sqrt{2\pi} \left(\frac{\sqrt{n} \log n}{\lambda}\right)^{\frac{\sqrt{n} \log n}{\lambda} - 1} \left(\frac{\sqrt{n} \log n}{\lambda}\right)^{\frac{1}{2}} e^{-\frac{\sqrt{n} \log n}{\lambda}} \left(1 + O\left(\frac{1}{\sqrt{n} \log n}\right)\right)} \\ &= \underbrace{\sqrt{n} \log n \left(\frac{n}{\sqrt{n} \log n}\right)^{\frac{\sqrt{n} \log n}{\lambda} - 1} e^{\frac{\sqrt{n} \log n - n}{\lambda}}}_{A_n} \underbrace{\frac{1 + O\left(\frac{\sqrt{n} \log n}{n}\right)}{\sqrt{2\pi} \left(\frac{\sqrt{n} \log n}{\lambda}\right)^{\frac{1}{2}} \left(1 + O\left(\frac{1}{\sqrt{n} \log n}\right)\right)}}_{B_n}. \end{aligned}$$

Clearly  $B_n \rightarrow 0$  as  $n \rightarrow \infty$  since only the numerator is bounded. We claim that  $A_n$  also approaches 0 as  $n \rightarrow \infty$ . Indeed, we can group the factors of  $A_n$  as

$$|A_n| = \left| e^{\frac{\sqrt{n} \log n - n}{\lambda}} (\sqrt{n})^{\frac{\sqrt{n} \log n}{\lambda}} \frac{\log n}{(\log n)^{\frac{\sqrt{n} \log n}{\lambda}}} \right| \leq \left| e^{\frac{\sqrt{n} \log n - n}{\lambda}} (\sqrt{n})^{\frac{\sqrt{n} \log n}{\lambda}} \right|$$

where we assumed that  $n$  was sufficiently large in the final inequality. At this point we can instead examine the limiting behavior of  $e^{\sqrt{n} \log n - n} (\sqrt{n})^{\sqrt{n} \log n}$  and infer the behavior of  $A_n$  by raising the result to the power  $\frac{1}{\lambda}$ . Writing  $\sqrt{n}$  as  $e^{\frac{1}{2} \log n}$  we have

$$(A_n)^\lambda = e^{\sqrt{n} \log n - n} (\sqrt{n})^{\sqrt{n} \log n} = e^{\sqrt{n} \log n + \frac{1}{2} \sqrt{n} \log n \log n - n}.$$

Since  $n$  grows faster than both  $\sqrt{n} \log n$  and  $\sqrt{n}(\log n)^2$ , it follows that this quantity approaches 0 as  $n \rightarrow \infty$  from which the desired result follows.  $\square$

With the above proposition proved, we set  $p = n - 1 - j$  and  $\tilde{q}_{p,n} = q_{j,n}$ , yielding

$$\frac{\log q_n}{\sqrt{n}} = \frac{1}{\sqrt{n}} \sum_{p=0}^{\sqrt{n} \log n} \log \tilde{q}_{p,n} + o(1). \quad (4.14)$$

Consider now the *Gamma distribution* which with  $X \sim \text{Gamma}(k, \theta)$  we recall has the properties

$$f_X(x) = \frac{1}{\Gamma(k)\theta^k} x^{k-1} e^{-\frac{x}{\theta}}, \quad \mathbb{E}(X) = k\theta, \quad \text{Var}(X) = k\theta^2$$

as can be found in most introductory books on probability such as [22]. If  $X \sim \text{Gamma}(k, \theta)$  with  $k = \frac{c+j+1}{\lambda} = \frac{c+n-p}{\lambda}$  and  $\theta = 1$ , we have

$$\tilde{q}_{p,n} = q_{j,n} = \mathbb{P}\left(X < \frac{n}{\lambda}\right).$$

To proceed we will need to approximate the above expression for large  $j$  as follows.

**Lemma 4.3.4.** *If  $X \sim \text{Gamma}(k, 1)$  and  $Y = \frac{X-k}{\sqrt{k}}$ , then  $Y$  converges to the standard normal as  $k \rightarrow \infty$ , i.e.*

$$\mathbb{P}(Y \leq a) \rightarrow \Phi(a) \quad \text{as } k \rightarrow \infty, \quad \Phi(a) = \frac{1}{2\pi} \int_{-\infty}^a e^{-t^2/2} dt$$

where  $\Phi$  is the standard normal cumulative distribution function.

In the proof of the above lemma we will make use the *characteristic function* of a random variable which we recall is the Fourier transform of its probability density function. This next theorem will allow us to infer the convergence of distributions from the convergence of characteristic functions.

**Theorem 4.3.5** (Lévy's continuity theorem). *Let  $\{X_n\}_{n=1}^{\infty}$  be a sequence of random variables such that the sequence of their characteristic functions,  $\{\chi_n\}_{n=1}^{\infty}$ , converges to  $\chi(t)$  for each  $t \in \mathbb{R}$ , where  $\chi$  is continuous at 0. Then there exists a random variable  $X$  with characteristic function  $\chi$  and  $X_n \rightarrow X$ .*

For a proof of the above theorem, see e.g. [37, Sec. 18.1]. It is worth mentioning that uniqueness of  $X$  follows from the injectivity of the Fourier transform.

*Proof of Lemma 4.3.4.* It is a well-known fact from Fourier analysis that the Fourier transform of a Gaussian, the standard normal probability density function, is another Gaussian so we want to show that the characteristic function of  $Y$  approaches  $e^{-t^2/2}$  as  $k \rightarrow \infty$ . We begin by noting that the characteristic function can be written as the expectation value of  $e^{itX}$ . With the substitution  $y = x(1-it)$  we have that

$$\begin{aligned} \chi_X(t) &= \mathbb{E}[e^{itX}] = \frac{1}{\Gamma(k)} \int_0^{\infty} e^{itx} x^{k-1} e^{-x} dx = \frac{1}{\Gamma(k)} \int_0^{\infty} \left(\frac{y}{1-it}\right)^{k-1} e^{-y} \frac{dy}{1-it} \\ &= \frac{1}{\Gamma(k)(1-it)^k} \int_0^{\infty} y^{k-1} e^{-y} dy = (1-it)^{-k}. \end{aligned}$$

Doing the same for  $Y$  we obtain

$$\chi_Y(t) = \mathbb{E} \left[ e^{it \frac{X-k}{\sqrt{k}}} \right] = e^{-i\sqrt{kt}} \chi_X \left( \frac{t}{\sqrt{k}} \right) = e^{-i\sqrt{kt}} \left( 1 - \frac{it}{\sqrt{k}} \right)^{-k}.$$

Knowing that our end goal is to write this as  $e^{-t^2/2}$ , we temporarily take the logarithm of  $(1 - it/\sqrt{k})^{-k}$  and consider the Taylor expansion.

$$\begin{aligned} \log \left[ \left( 1 - \frac{it}{\sqrt{k}} \right)^{-k} \right] &= -k \log \left[ 1 - \frac{it}{\sqrt{k}} \right] \\ &= -k \left[ \frac{-it}{\sqrt{k}} - \frac{1}{2} \left( \frac{it}{\sqrt{k}} \right)^2 + O \left( \frac{1}{k^{3/2}} \right) \right] = i\sqrt{kt} - \frac{t^2}{2} + O \left( \frac{1}{\sqrt{k}} \right) \implies \\ \chi_Y(t) &= e^{-i\sqrt{kt}} e^{i\sqrt{kt} - t^2/2 + O(\frac{1}{\sqrt{k}})} = e^{-t^2/2 + O(\frac{1}{\sqrt{k}})} \xrightarrow[k \rightarrow \infty]{} e^{-t^2/2}. \end{aligned}$$

With the pointwise convergence established, the desired result follows from Lévy's continuity theorem.  $\square$

In view of (4.14) we can now write

$$\begin{aligned} \frac{\log q_n}{\sqrt{n}} &= \frac{1}{\sqrt{n}} \sum_{p=0}^{\sqrt{n} \log n} \log \left[ \mathbb{P} \left( X < \frac{n}{\lambda} \right) \right] + o(1) \xrightarrow[n \rightarrow \infty]{} \frac{1}{\sqrt{n}} \sum_{p=0}^{\sqrt{n} \log n} \log \left[ \Phi \left( \frac{p}{\sqrt{\lambda n}} \right) \right], \quad (4.15) \\ \Phi(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt = \frac{1}{2} \left[ 1 + \operatorname{erf} \left( \frac{t}{\sqrt{2}} \right) \right] \end{aligned}$$

where  $\Phi$  is the standard normal cumulative distribution function. This since

$$\mathbb{P} \left( X < \frac{n}{\lambda} \right) = \mathbb{P} \left( \frac{X-k}{\sqrt{k}} < \frac{\frac{n}{\lambda} - k}{\sqrt{k}} \right) = \mathbb{P} \left( Y < \frac{p-c}{\lambda \sqrt{\frac{n-p+c}{\lambda}}} \right) \xrightarrow[n \rightarrow \infty]{} \mathbb{P} \left( Y < \frac{p}{\sqrt{\lambda n}} \right)$$

where we in the last step used that the  $p < \sqrt{n} \log n$  condition yields that  $\frac{n-p+c}{\lambda} \rightarrow \frac{n}{\lambda}$  and  $\frac{c}{\sqrt{n}} \rightarrow 0$  as  $n \rightarrow \infty$ .

We now rewrite (4.15) as a Riemann sum. Since the argument is zero for  $p = 0$  and goes to infinity as  $n \rightarrow \infty$  for  $p = \sqrt{n} \log n$  we want to integrate from 0 to  $\infty$ . Setting  $\xi_p = \frac{p}{\sqrt{n}}$  and noting that  $\Delta \xi = \frac{p+1}{\sqrt{n}} - \frac{p}{\sqrt{n}} = \frac{1}{\sqrt{n}}$  we have that as  $n \rightarrow \infty$ , (4.15) becomes

$$\sum_{p=0}^{\sqrt{n} \log n} \log \left[ \Phi \left( \frac{\xi_p}{\sqrt{\lambda}} \right) \right] \Delta \xi \rightarrow \int_0^\infty \log \left[ \Phi \left( \frac{t}{\sqrt{\lambda}} \right) \right] dt = \sqrt{\lambda} \int_0^\infty \log [\Phi(u)] du =: -\kappa \sqrt{\lambda}.$$

This integral does not appear to be solvable with any common CAS such as Maple or Mathematica but is much faster to compute to high precision than (4.10). Doing so we obtain  $\kappa \approx 0.477\ 535\ 339\ 808$ . Returning to the partition function we have

$$\begin{aligned} \log q_n &= \log Z_n^{\text{hard}} - \log Z_n^{\text{free}} = -\kappa \sqrt{\lambda} \sqrt{n} + o(1) \implies \\ -\log Z_n^{\text{hard}} &= n^2 \left[ \frac{3}{4\lambda} + \frac{\log \lambda}{2\lambda} \right] - \frac{n}{2} \log n \\ &\quad + n \left[ 1 - \frac{\log(2\pi)}{2} + \frac{1+2c-\lambda+(1+2c+\lambda)\log \lambda}{2\lambda} \right] + \kappa \sqrt{\lambda} \sqrt{n} + O(\log n). \end{aligned}$$

It is interesting to note that  $\kappa\sqrt{\lambda}$  is the only term which scales as  $\sqrt{n}$ . The fact that this is the case for all  $\lambda$  suggests the following conjecture.

**Conjecture 4.3.6.** *Let  $Z_n$  denote the partition function of any ensemble. With notation as above we have that*

$$\log \left( \frac{Z_n^{\text{hard}}}{Z_n^{\text{free}}} \right) = C\sqrt{n} + o(1) \quad \text{as } n \rightarrow \infty$$

where  $C$  depends only on  $Q$ .

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Bachelor's Theses in Mathematical Sciences 2019:K7  
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