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Quantum harmonic analysis on locally compact groups

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What and why of quantum harmonic analysis

Quantum harmonic analysis is:
 - A framework where convolutions between functions and operators and operators and operators are defined.

We care about it because:
 - It provides a new lens to view classical objects.

Our contribution:
 - Extracting the essence of quantum harmonic analysis to extend it to arbitrary locally compact groups.

"Classical" quantum harmonic analysis

Roots in quantum physics and time-frequency analysis, defined using the following representation of the Weyl-Heisenberg group

$$\pi(x,\omega)\psi(t) = e^{2\pi i\omega\cdot t}\psi(t-x)$$

"Time-frequency shift"

here $\pi : \mathbb{R}^{2d} \to \mathcal{U}(L^2(\mathbb{R}^d))$ is a square integrable representation.

These operators are central in time-frequency analysis

$$V_\varphi\psi(x,\omega) = \langle \psi, \boldsymbol{\pi(x,\omega)}\varphi\rangle = \int_{\mathbb{R}^d} \psi(t)e^{-2\pi i\omega\cdot t}\overline{\varphi(t-x)}\,dt.$$

New convolutions

Regular convolutions are defined as

$$f * g(x) = \int_{\mathbb{R}} f(y) T_{-y} g(x)\, dy$$

to get this for operators, we need to be able to integrate and translate operators

$$T_z f \to \alpha_z(S) = \pi(z)^* S \pi(z),$$
$$\int f \to \operatorname{tr}(S).$$

This gives rise to the following definitions:

$$f \star S = \int_{\mathbb{R}^{2d}} f(z) \alpha_z(S)\, dz \qquad \text{(function} \star \text{operator} = \text{operator)},$$
$$T \star S(z) = \operatorname{tr}(T \alpha_z(S)) \qquad \text{(operator} \star \text{operator} = \text{function)}.$$

Convolution properties

Boundedness:

$$\|f \star S\|_{\mathcal{S}^p} \leq \begin{matrix} \|f\|_{L^1}\|S\|_{\mathcal{S}^p}, \\ \|f\|_{L^p}\|S\|_{\mathcal{S}^1}, \end{matrix}$$

$$\|T \star S\|_{L^p} \leq \|T\|_{\mathcal{S}^p}\|S\|_{\mathcal{S}^1}.$$

Associativity:

$$(f \star S) \star T(z) = f * (S \star T)(z),$$
$$(f * g) \star S = f \star (g \star S).$$

Adjoints:

$$\mathcal{A}_S : L^p(G) \to \mathcal{S}^p, \qquad f \mapsto f \star S,$$
$$\mathcal{B}_S : \mathcal{S}^p \to L^p(G), \qquad T \mapsto T \star S,$$
$$\mathcal{A}_S^* = \mathcal{B}_S.$$

Fourier:

$$\mathcal{F}_W(S)(z) = \mathrm{tr}(\pi(-z)S)$$
$$\mathcal{F}_W(f \star S) = \mathcal{F}_\sigma(f) \cdot \mathcal{F}_W(S),$$
$$\mathcal{F}_W(T \star S)(z) = \mathcal{F}_W(T)(z) \cdot \mathcal{F}_W(S)(z).$$

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What we get from quantum harmonic analysis

- Function-operator convolutions coincide with localization operators from time-frequency analysis and are unitarily equivalent to Gabor-Toeplitz operators.
- Operator-operator convolutions coincide with Cohen's class of time-frequency distributions which generalize the classical spectrogram.

This makes several problems easier to approach:

- Cohen phase retrieval uniqueness
- Symbol recovery for localization operators
- Compactness characterization of Gabor-Toeplitz operators
- Analysis of convolutional neural networks

Locally compact setting

What do we change in the locally compact setting?

- Weyl-Heisenberg $\mathbb{R}^{2d} \to$ locally compact group G,
- Time-frequency shift $\pi \to$ square integrable representation σ of G,
- Signals in $L^2(\mathbb{R}^d) \to$ Hilbert space $\mathcal{H}$,
- Lebesgue measure $dz \to$ (right) Haar measure $d\mu_R$.

Why do we care about the locally compact setting?

- Affine group
- Similitude group
- Shearlet group
- Affine Poincaré group

New(er) convolutions

$$f \star_G S = \int_G f(x)\sigma(x)^* S\sigma(x)\, d\mu_R(x), \qquad T \star_G S(x) = \operatorname{tr}\left(T\sigma(x)^* S\sigma(x)\right).$$

Admissibility of operators

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Integrability of operator-operator convolutions is natural and can be deduced as a consequence of the Duflo-Moore orthogonality relation

$$\int_G \langle \psi_1, \sigma(x)^* \phi_1 \rangle \overline{\langle \psi_2, \sigma(x)^* \phi_2 \rangle} \, d\mu_R(x) = \langle \psi_1, \psi_2 \rangle \overline{\langle \mathcal{D}^{-1} \phi_1, \mathcal{D}^{-1} \phi_2 \rangle}$$

$$\implies \int_G T \star_G S(x) d\mu_R(x) = \operatorname{tr}(T) \operatorname{tr}(\mathcal{D}^{-1} S \mathcal{D}^{-1}).$$

We say that S for which this is finite are **admissible**.

Admissibility characterizes nice mappings

Theorem (Kiukas et al.)

Suppose $\Gamma : L^\infty(G) \to B(\mathcal{H})$ *satisfies*

1. *Positive functions* $\mapsto$ *positive operators,*
2. $1 \mapsto I_{\mathcal{H}}$,
3. *Weak* - weak* continuous,*
4. $\sigma(x)^*\Gamma(f)\sigma(x) = \Gamma(f(\cdot x^{-1}))$.

Then

$$\Gamma(f) = f \star_G S$$

where S *is admissible.*

New mapping properties

The mapping bounds now look like

$$\|f \star_G S\|_{\mathcal{S}^p} \leq \begin{array}{l} \|f\|_{L^1_r(G)} \|S\|_{\mathcal{S}^p}, \\ \|f\|_{L^p_r(G)} \|S\|_{\mathcal{S}^1}^{1/p} \|\mathcal{D}^{-1} S \mathcal{D}^{-1}\|_{\mathcal{S}^1}^{1/q}, \end{array}$$

$$\|T \star_G S\|_{L^\infty(G)} \leq \|S\|_{\mathcal{S}^p} \|T\|_{\mathcal{S}^q},$$

$$\|T \star_G S\|_{L^p_r(G)} \leq \|T\|_{\mathcal{S}^p} \|S\|_{\mathcal{S}^1}^{1/q} \|\mathcal{D}^{-1} S \mathcal{D}^{-1}\|_{\mathcal{S}^1}^{1/p}.$$

The same mappings

$$\mathcal{A}_S : L^p_r(G) \to \mathcal{S}^p, \qquad f \mapsto f \star_G S,$$
$$\mathcal{B}_S : \mathcal{S}^p \to L^p_r(G), \qquad T \mapsto T \star_G S$$

are only adjoints when S is admissible.

Wiener's Tauberian theorem

We say a function or operator is p-regular if

$$\overline{\operatorname{span}\left\{g(\cdot x^{-1})\right\}_{x\in G}} = L^p_r(G), \qquad \overline{\operatorname{span}\left\{\sigma(x)^* S \sigma(x)\right\}_{x\in G}} = \mathcal{S}^p.$$

The following are equivalent:

1. S is p-regular,
2. If $f \in L^q_r(G)$ and $f \star_G S = 0$, then $f = 0$,
3. $\mathcal{S}^p \star_G S$ is dense in $L^p_r(G)$,
4. If $T \in \mathcal{S}^q$ and $T \star_G S = 0$, then $T = 0$,
5. $L^p_r(G) \star_G S$ is dense in $\mathcal{S}^p$,
6. $S \star_G S$ is p-regular,
7. For any regular $T \in \mathcal{S}^1, T \star_G S$ is p-regular.

Berezin-Lieb inequalities

Theorem

Let Φ be convex and S admissible with $\mathrm{tr}(\mathcal{D}^{-1}S\mathcal{D}^{-1}) = 1$, then

$$\int_G \Phi \circ (T \star_G S)(x) d\mu_R(x) \leq \frac{\mathrm{tr}\left(\Phi(\mathrm{tr}(S)T\right)}{\mathrm{tr}(S)},$$

$$\mathrm{tr}(\Phi(f \star_G S)) \leq \mathrm{tr}(S) \int_G \Phi\big(f(x)\big) d\mu_R(x).$$

If $\Phi = \mathrm{Id}$ we have equality.

Basically Jensen's inequality for convex functions but for functions and operators!

Quantization on exponential groups (ongoing work)

Quantization: A (nice) mapping from functions to operators

$$L^2(G) \to \mathcal{HS}, \ f \mapsto A_f$$

The inverse mapping is denoted by $S \mapsto a_S$. We have

$$\begin{aligned} A_{f*g} &= f \star_G A_g, \\ A_f \star A_g &= f * \check{g}, \\ A_{T \star S} &= a_T \star \check{S}. \end{aligned}$$

Thanks for your attention!

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