NTNU

Norwegian University of Science and Technology

Symbol recovery for localization operators

A quantum harmonic analysis approach

Simon Halvdansson

Hannover, October 4, 2022

Time-frequency analysis and localization operators

In time-frequency analysis, a central object is the *short-time Fourier transform* $V_g : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d})$

$$V_g\psi(x,\omega) = \int_{\mathbb{R}^d} \psi(t)\overline{g(t-x)}e^{-2\pi i t\cdot\omega}\,dt.$$

From it, we can reconstruct ψ as

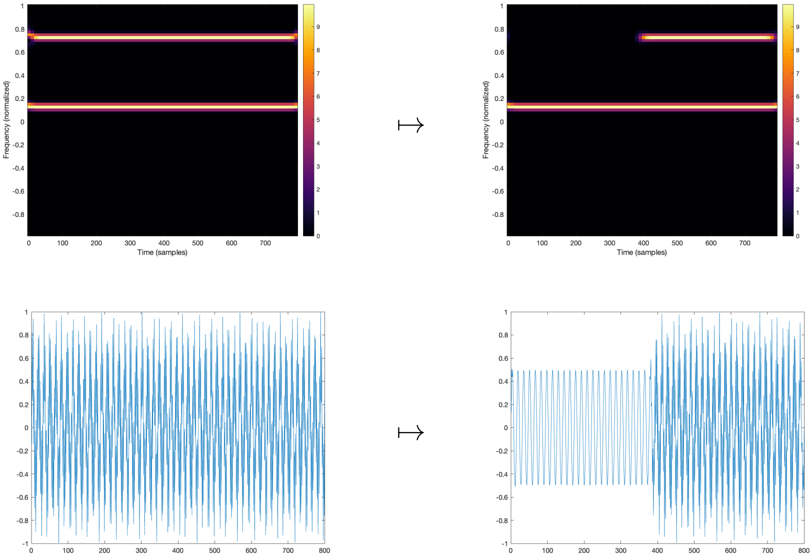
$$\psi = \int_{\mathbb{R}^{2d}} V_g\psi(z)\pi(z)g\,dz.$$

A *localization operator* is constructed by weighing this reconstruction with a symbol $f : \mathbb{R}^{2d} \to \mathbb{R}$

$$A_f^g\psi = \int_{\mathbb{R}^{2d}} f(z)V_g\psi(z)\pi(z)g\,dz.$$

NTNU

Example of localization operator action



The (inverse) problem

Given some information about A_f^g, estimate the symbol f

Previously investigated by:

- Abreu and Dörfler (2012),
- Abreu, Gröchenig and Romero (2014),
- Luef and Skrettingland (2018),
- Romero and Speckbacher (2022)

Four approaches:

- Fourier approach
- Look at spectral data of A_f^g
- Apply A_f^g to white noise
- Tiling the TF plane

Quantum harmonic analysis crash course I

NTNU

- Function-operator convolutions:

$$f \star S = \int_{\mathbb{R}^{2d}} f(z)\pi(z)S\pi(z)^* \, dz, \qquad f \star (g \otimes g) = A_f^g.$$

- Operator-operator convolutions:

$$T \star S(z) = \operatorname{tr}\left(T\pi(z)S\pi(z)^*\right), \qquad (\psi \otimes \psi) \star (\varphi \otimes \varphi)(z) = |V_\varphi \psi(z)|^2.$$

- Fourier-Wigner transform:

$$\mathcal{F}_W(S) = e^{-\pi i x \cdot \omega} \operatorname{tr}(\pi(-z)S), \qquad \mathcal{F}_W(\varphi \otimes \varphi)(z) = e^{\pi i x \cdot \omega} V_\varphi \varphi(z).$$

Quantum harmonic analysis crash course II

Boundedness:

$$\|f \star S\|_{\mathcal{S}^p} \leq \begin{matrix} \|f\|_{L^1}\|S\|_{\mathcal{S}^p}, \\ \|f\|_{L^p}\|S\|_{\mathcal{S}^1}, \end{matrix}$$

$$\|T \star S\|_{L^p} \leq \|T\|_{\mathcal{S}^p}\|S\|_{\mathcal{S}^1}.$$

Associativity:

$$(f \star S) \star T(z) = f * (S \star T)(z),$$

$$(f * g) \star S = f \star (g \star S).$$

Adjoints:

$$\mathcal{A}_S : L^p(G) \to \mathcal{S}^p, \qquad f \mapsto f \star S,$$

$$\mathcal{B}_S : \mathcal{S}^p \to L^p(G), \qquad T \mapsto T \star S,$$

$$\mathcal{A}_S^* = \mathcal{B}_S.$$

Fourier:

$$\mathcal{F}_W(f \star S) = \mathcal{F}_\sigma(f) \cdot \mathcal{F}_W(S),$$

$$\mathcal{F}_W(T \star S)(z) = \mathcal{F}_W(T)(z) \cdot \mathcal{F}_W(S)(z).$$

Symbol recovery = QHA deconvolution

- An equivalent view of symbol recovery is as deconvolution of function-operator convolutions.
- This is interesting in its own right as a problem in quantum harmonic analysis.
- Also poses the question of inverting $f \mapsto f \star S$ for $S \in \mathcal{S}^1$ (symbol recovery for mixed-state localization operators).

Uniqueness of problem is similar to phase retrieval:

$$f \mapsto f \star S \text{ injective} \iff \mathcal{F}_W(S) \neq 0.$$

Fourier deconvolution

We can try to apply a convolution theorem directly to disentangle the function-operator convolution.

$$\begin{aligned}\mathcal{F}_W(A_f^g)(z) &= \mathcal{F}_W(f \star (g \otimes g))(z) = \mathcal{F}_\sigma(f)(z)\mathcal{F}_W(g \otimes g)(z) \\ &= \mathcal{F}_\sigma(f)(z)A(g)(z) = \mathcal{F}_\sigma\Big(f * W(g)\Big)(z).\end{aligned}$$

$$\implies \mathcal{F}_\sigma(\mathcal{F}_W(A_f^g)) = f * W(g)$$

This can be deconvolved!

- Requires full spectral knowledge (to compute $\mathcal{F}_W(A_f^g)$!)
- Requires knowledge of window / blind deconvolution

(We have computed the Weyl symbol of A_f^g)

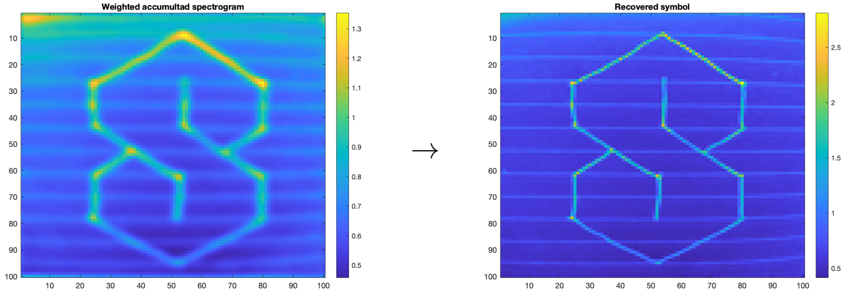
Spectral approach

Convolve with $\varphi \otimes \varphi$:

$$A_f^g \star (\varphi \otimes \varphi)(z) = f * (g \otimes g) \star (\varphi \otimes \varphi)(z) = f * |V_g\varphi|^2(z)$$

$$= \operatorname{tr}\left(A_f^g \pi(z)(\varphi \otimes \varphi)\pi(z)^*\right) = \left(\sum_k \lambda_k h_k \otimes h_k\right) \star (\varphi \otimes \varphi)(z) = \sum_k \lambda_k |V_\varphi h_k(z)|^2$$

If we know g, this turns into a deconvolution problem:



Spectral approach - pure operator formulation

Taking the viewpoint of inverting $f \mapsto f \star S$, we can make this approach a bit clearer:

$$(f \star S) \star S = f * (S \star S).$$

If we don't know S, we can make a guess:

$$(f \star S) \star T = f * (S \star T).$$

The closer T is to S, the more "Gaussian-like" $S \star T$ will be.

Spectral approach - result formulation

Theorem

Let $f \in L^1(\mathbb{R}^{2d})$ be real-valued and of bounded variation and $S \in \mathcal{S}^1$ be positive with $(S) = 1$. Then if $f \star S = \sum_k \lambda_k h_k \otimes h_k$,

$$\left\| \sum_{k=1}^{N} \lambda_k Q_S(h_k) - f \right\|_{L^1} \leq \sum_{k=N+1}^{\infty} |\lambda_k| + \mathrm{Var}(f) \int_{\mathbb{R}^{2d}} |z|(S \star S)(z)\, dz.$$

Moreover, in the $N = \infty$ case,

$$\sum_{k=1}^{\infty} \lambda_k Q_S(h_k)(z) = f * (S \star S)(z)$$

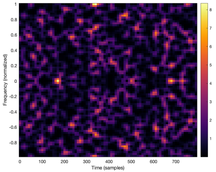
which can be deconvolved in the sense that

$$f = \mathcal{F}^{-1} \left(\frac{\mathcal{F}(f * (S \star S))}{\mathcal{F}(S \star S)} \right).$$

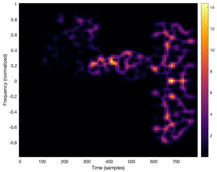
NTNU

White noise approach

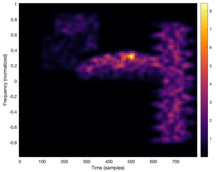
Idea: Spectrogram of white noise is *approximately* uniform



Intuitively: Applying localization operator to white noise should hence weigh this based on f



Improvement: To get rid of noise, take the average over many realizations

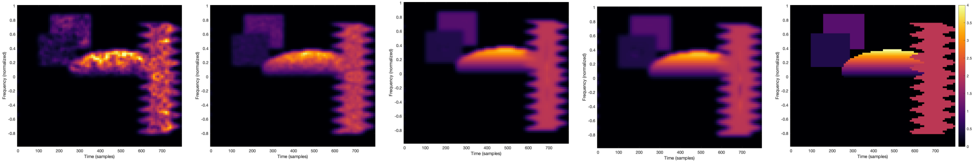


White noise estimator

Formally and visually, what does this look like?

$$\rho(z) = \frac{1}{K}\sum_{k=1}^{K} V_\varphi(A_f^g \mathcal{N}_k) \approx \sum_m \lambda_m^2 |V_\varphi h_m(z)|^2 \approx f^2 * |V_\varphi g|^2(z) \approx f(z)^2.$$

ρ_{20} ρ_{200} ϑ $f^2 * |V_g g|^2$ f^2



Error estimation:

$$\sum_m \lambda_m^2 |V_\varphi h_m(z)|^2 = \left(A_f^g\right)^2 \star (\varphi \otimes \varphi)(z) = \left(A_{f^2}^g + \text{Error}\right) \star (\varphi \otimes \varphi)$$

$$= f^2 * |V_\varphi g|^2 + \text{Error} * (\varphi \otimes \varphi)$$

White noise L^1 error

Theorem
Let $f \in C_c^{d+2}(\mathbb{R}^{2d})$, ρ be given by $\rho(z) = \frac{1}{K}\sum_{k=1}^{K} |V_\varphi(A_f^g \mathcal{N}_k)(z)|^2$ with white noise variance σ^2, $g, \varphi \in \mathcal{S}(\mathbb{R}^d)$ with $\|g\|_{L^2} = \|\varphi\|_{L^2} = 1$ and define

$$B_1 = A\left[\|K\|_{L^2} + \left(\sum_{j=1}^{2d} \left\|\partial_j^{d+1} K\right\|_{L^2}^2\right)^{1/2}\right], \quad B_2 = \left(\int_{\mathbb{R}^{2d}} \left|(\nabla f^2)(z)\right| dz\right)\left(\int_{\mathbb{R}^{2d}} |z||V_\varphi g(z)|^2\, dz\right)$$

where

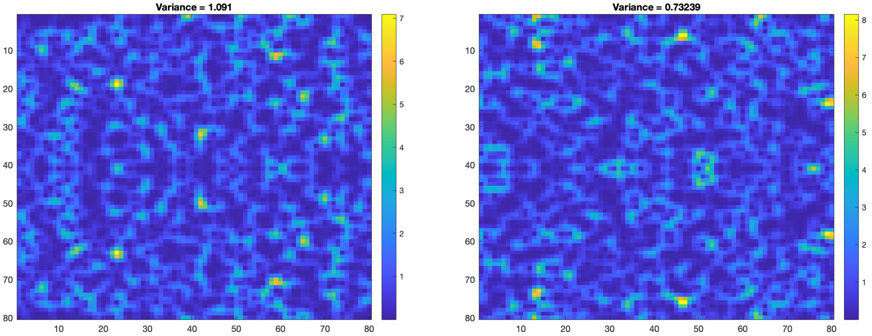
$$K(y,z) = f(y)\left(\sum_{|\alpha|=1} \int_0^1 \partial^\alpha f(y + t(z-y))\, dt (z-y)\right) V_\varphi \varphi(y-z)$$

and A is a constant independent of K. Then there exists a $C \leq 1$ such that

$$\mathbb{P}\left(\int_{\mathbb{R}^{2d}} \left|\frac{\rho(z)}{\sigma^2} - f(z)^2\right| dz > B_1 + B_2 + t\right) \leq \frac{\|f\|_{L^2}^2}{t\sqrt{K}}\left[\frac{3}{2\sqrt{\pi C}} \operatorname{erf}\left(\sqrt{CK}\right) + \frac{3}{C\sqrt{K}} e^{-CK}\right].$$

"Optimal" white noise

If we have control over the input, we could choose "optimal" white noise with less unevenness.



Taking this to its extreme conclusion would mean a "flat" spectrogram.

Plane tiling

Idea: With white noise, we filled the time-frequency plane with white noise - let's instead fill it by an orthonormal basis!

$$\sum_n |V_\varphi e_n(z)|^2 = I \star (\varphi \otimes \varphi)(z) = (1 \star \varphi \otimes \varphi) \star (\varphi \otimes \varphi)(z) = 1 * |V_\varphi \varphi|^2(z) = 1.$$

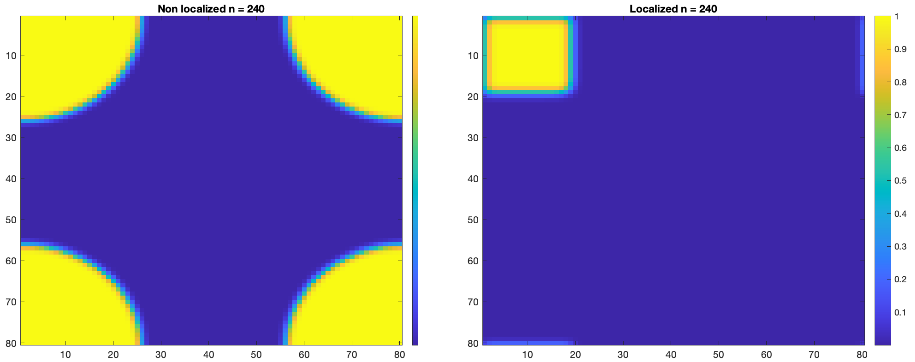
By replacing $\{e_n\}_n$ by $\left\{A_f^g e_n\right\}_n$ this tiling will be weighed by f^2:

$$\begin{aligned}
\sum_n |V_\varphi(A_f^g e_n)(z)|^2 &= \sum_n (A_f^g e_n \otimes A_f^g e_n) \star (\varphi \otimes \varphi)(z) \\
&= A_f^g \left(\sum_n e_n \otimes e_n \right) A_f^g \star (\varphi \otimes \varphi)(z) \\
&= (A_f^g I A_f^g) \star (\varphi \otimes \varphi)(z) \\
&= (A_f^g)^2 \star (\varphi \otimes \varphi)(z) \approx A_{f^2}^g \star (\varphi \otimes \varphi)(z) \approx f(z)^2.
\end{aligned}$$

Let's try it out!

NTNU

Plane tiling example



Frame tiling limitations

This method is expensive if we need many terms! Hence it should only be used when we know something about the support of f, then the inputs can be chosen to cover this area in the time-frequency plane.

We don’t have to use an ONB though - when we have full control over input, it can be replaced by well chosen uniform white noise.

Summary of methods (rank one)

$$
\begin{aligned}
\mathcal{F}_\sigma(\mathcal{F}_W(f \star (g \otimes g))) &= f * W(g)(z) \approx f(z) \\
\big(f \star (g \otimes g)\big) \star \varphi \otimes \varphi(z) &= f * |V_\varphi g|^2(z) \approx f(z) \\
\begin{matrix} |V_\varphi(A_f^g(\mathcal{N}))|^2 \\ \sum_n |V_\varphi(A_f^g e_n)(z)|^2 \end{matrix} &\approx f^2 * |V_\varphi g|^2(z) \approx f(z)^2
\end{aligned}
$$

Takeaway:
Quantum harmonic analysis provides an appropriate framework to study localization operators!

Future work

Mixed-state localization operators ($f \star S$):

- Requires asymptotics of products $(f_1 \star S_1)(f_2 \star S_2)$
- New results on $\frac{1}{K}\sum_{k=1}^{K} Q_S(f \star S(\mathcal{N}_k)) \xrightarrow{k\to\infty} \sum_m \lambda_m^2 Q_S(h_m)$

Replacing white noise by more general "ambient" input

- If $|V_\varphi(\mathcal{N})|^2 \sim \rho$, then $|V_\varphi(A_f^g \mathcal{N})|^2 \sim f^2 \cdot \rho$

Replace white noise by optimal input

- If we can find $U \in L^2(\mathbb{R}^d)$ such that $|V_\varphi(U)|^2 \approx 1$ on some large ball, then $|V_\varphi(A_f^g U)|^2$ should approximate f^2 on that ball.

Kernel methods

- If we can estimate the integral kernel of A_f^g, we can get f from it.

Toeplitz interpretation

- Localization operators are also (Fock) Toeplitz operators, are the results interesting in this context?

NTNU

Thank you!