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Existence and Approximations of Wavelet Uncertainty Minimizers

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September 14, 2021

Based on joint work with Ron Levie, Jan-Fredrik Olsen and Nir Sochen

Continuous wavelet transform

The wavelet transform $W_f : L^2(\mathbb{R}) \to L^2(\mathbb{R}^2, e^{-\alpha})$

$$W_f[s](\alpha, \beta) = e^{-\alpha/2} \int_{\mathbb{R}} s(t) \overline{f\left(\frac{t-\beta}{e^{\alpha}}\right)} \, dt$$

is induced by the **representation**

$$\pi(\alpha, \beta) f(t) = e^{-\alpha/2} f\left(\frac{t-\beta}{e^{\alpha}}\right)$$

of **time and scale shifts** in the sense that

$$W_f[s](\alpha, \beta) = \langle s, \pi(\alpha, \beta) f \rangle_{L^2}$$

Examples of wavelets

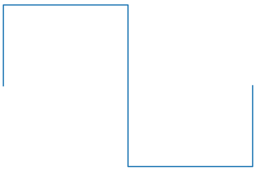

Figure: Haar

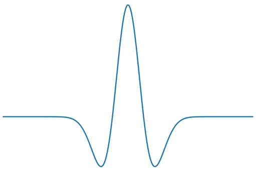

Figure: Mexican hat

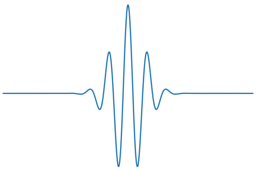

Figure: Morlet

Figure: Coiflet

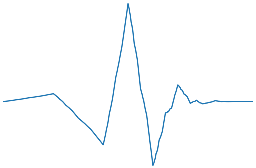

Figure: Daubechies

Figure: Symlet

An example (2 chirps)

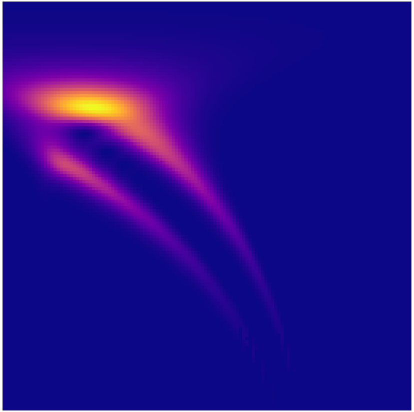

Figure: Wavelet transform of two chirps with respect to Morlet wavelet

Wavelet design

Mother wavelet f must only satisfy the **admissibility condition**

$$\int_0^\infty \frac{|\hat{f}(\omega)|^2}{\omega} d\omega < \infty$$

to guarantee invertibility

Wavelet design: Constructing mother wavelets f with desirable properties for the continuous wavelet transform

Wavelet uncertainty

Want wavelet transform to be well localized in $\mathbb{R}^2$

$\rightarrow$ Want wavelet to be well localized in **time** and **scale**

- Historically, wavelet design has been led by qualitative features[1]
- Quantiative approaches have previously been unsuccessful[2,3]
- A framework by Levie and Sochen enables a quantative approach[4]

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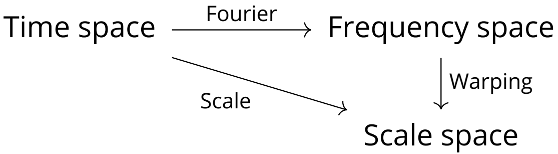
1 Daubechies, *Ten lectures on wavelets*, (1992)
2 Dahlke and Maass, *The affine uncertainty principle in one and two dimensions*, (1995)
3 Maass et al, *Do uncertainty minimizers attain minimal uncertainty?*, (2010)
4 Levie and Sochen, *Uncertainty principles and optimally sparse wavelet transforms*, (2020)

Localization

To talk about scale, we introduce **scale space**:



Elements of the scale space are denoted as $\tilde{f}$,

$$\tilde{f}(\sigma) = e^{-\sigma/2}\hat{f}(e^{-\sigma})$$

Localization

To measure how localized a wavelet is in time and scale, we introduce the **observables**:

$$T_x : f(t) \mapsto t f(t), \qquad \tilde{T}_\sigma : \tilde{f}(\sigma) \mapsto \sigma \tilde{f}(\sigma)$$

We measure localization with **expected value** and **variance**:

$$e_f(T) = \left\langle Tf, f \right\rangle, \qquad v_f(T) = \left\langle (T - e_f(T))^2 f, f \right\rangle$$

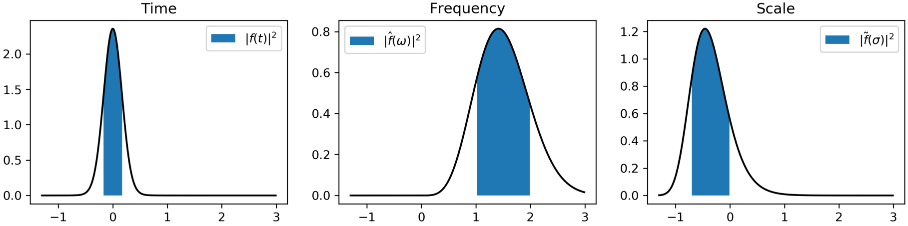


Figure: The complex Mexican hat wavelet

Localization

The observables T_x, T_σ are canonical (wrt. π) in the sense that

$$e_{\pi(0,\beta)f}(T_x) = e_f(T_x) + \beta,$$
$$e_{\pi(\alpha,0)f}(T_\sigma) = e_f(T_\sigma) + \alpha$$

Signal space uncertainty

Makes sense to define uncertainty as variance of properties we measure

$$\mathcal{L}_{\mathrm{S}}(f) := e^{-2e_f(T_\sigma)} v_f(T_x) + v_f(T_\sigma)$$

- Want **invariance** $\mathcal{L}_{\mathrm{S}}(f) = \mathcal{L}_{\mathrm{S}}\big(\pi(\alpha, \beta) f\big)$

Lemma

If $e_f(T_x) = e_f(T_\sigma) = 0$*,*

$$\mathcal{L}_{\mathrm{S}}(f) = \left\|T_x f\right\|^2 + \left\|T_\sigma f\right\|^2$$

Phase space uncertainty

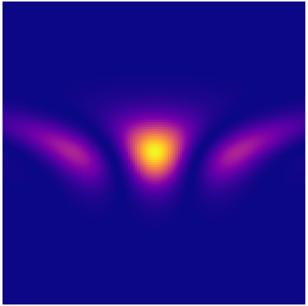
The **phase space uncertainty** was introduced by Levie, Avraham and Sochen via the ambiguity function $K_f(\alpha,\beta) = \left\langle f, \pi(\alpha,\beta) f \right\rangle$

Related to the "blurriness" of the wavelet transform by

$$Q \in W_f[L^2] \implies Q = K_f * Q$$

Related observables are:

$$A : F(\alpha,\beta) \mapsto \alpha F(\alpha,\beta),$$
$$B : F(\alpha,\beta) \mapsto \beta F(\alpha,\beta)$$



Phase space uncertainty

Definition

$$\mathcal{L}_{\mathrm{P}}(f) = v_{K_f}(A) + v_{K_f}(B)$$

Using wavelet-Plancherel theory[5]:

Theorem

If $e_f(T_x) = e_f(T_\sigma) = 0$*,*

$$\mathcal{L}_{\mathrm{P}}(f) = \underbrace{\left\|T_x f\right\|^2 + \left\|T_\sigma f\right\|^2}_{\textit{from } \mathcal{L}_{\mathrm{S}}(f)} + v^{\mathcal{W}}_{\frac{\hat{f}}{\|\hat{f}\|_{\mathcal{W}}}}\left(i\omega\frac{\partial}{\partial\omega}\right)\left\|\frac{\hat{f}}{\omega}\right\|^2 + v^{\mathcal{W}}_{\frac{\hat{f}}{\|\hat{f}\|_{\mathcal{W}}}}(-\ln(\omega))$$

[5] Levie and Sochen, *A wavelet plancherel theory with application to sparse continuous wavelet transform*, (2017)

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Existence

Theorem
Let $\mathcal{L}$ be one of $\mathcal{L}_{\mathrm{S}}$ and $\mathcal{L}_{\mathrm{P}}$. Then there exists an $f \in L^2$ with $e_f(T_x) = e_f(T_\sigma) = 0$ such that

$$\mathcal{L}(f) = \inf_{y \in \mathcal{D}} \mathcal{L}(y)$$

where $\mathcal{D}$ is the domain of $\mathcal{L}$

S. Halvdansson, J-F. Olsen, N. Sochen, and R. Levie, *Existence of wavelet uncertainty minimizers*, (2021)

Sketch of proof of existence

Consider a minimizing sequence $(f_n)_n \subset \mathcal{D}_{\mathrm{S}}$ (with $e_{f_n}(T_x) = e_{f_n}(T_\sigma) = 0$) i.e.

$$\mathcal{L}_{\mathrm{S}}(f_n) \xrightarrow[n\to\infty]{} \inf_{y\in\mathcal{D}_{\mathrm{S}}} \mathcal{L}_{\mathrm{S}}(y)$$

For large enough n, $f_n \in \mathcal{K}$ where

$$\mathcal{K} = \left\{ f \in \mathcal{D}_{\mathrm{S}} : \|T_x f\|^2 \leq K,\ \|T_\sigma f\|^2 \leq K \right\}$$

for some constant K

Strategy:

$\mathcal{K}$ compact $\implies$ existence of a minimizer

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Sketch of proof of existence

Goal: Show $\mathcal{K}$ compact

$\mathcal{K}$ closed:
Write $\mathcal{K}$ as intersection of closed subsets.

$$\{e_f(T_x) = 0, \|T_x f\|^2 \leq K\}$$
$$\{e_f(T_\sigma) = 0, \|T_\sigma f\|^2 \leq K\}$$

$\mathcal{K}$ pre-compact:

Lemma

For any $\varepsilon > 0$, there exists a compact subset $C_{a,b}$ of L^2 such that for any $f \in \mathcal{K}$, there is a $y \in C_{a,b}$ such that

$$\|f - y\| < \varepsilon.$$

Lemma $\implies$ any sequence in $\mathcal{K}$ has a Cauchy subsequence

L^2 complete $\implies$ $\mathcal{K}$ pre-compact $\square$

Smoothness properties

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Theorem

If f is a minimizer of $\mathcal{L}_{\mathrm{S}}$ with $e_f(T_x), e_f(T_\sigma) = 0$, then $f \in C^\infty$ and

$$f''(\omega) = \Big(\ln(\omega)^2 + \boldsymbol{\kappa} \ln(\omega) - \boldsymbol{\sigma} \Big) f(\omega)$$

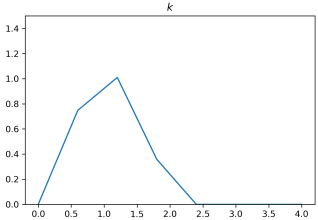
where

$$\begin{aligned} \boldsymbol{\kappa} &= 2\|f'\|^2, \\ \boldsymbol{\sigma} &= \|f'\|^2 + \|\ln(\omega) f\|^2 \end{aligned}$$

Why do we need to approximate?

We can only optimize functions in finite dimensional spaces
$\rightarrow$ **Space of linear splines** with spacing h

$$P_h(0,b) \subset L^2(0,b)$$



We want to verify:

$$\inf_{y \in \boldsymbol{P_h(0,b)}} \mathcal{L}(y) \xrightarrow[b \to \infty]{h \to 0} \inf_{y \in \boldsymbol{\mathcal{D}}} \mathcal{L}(y),$$

$$\operatorname*{argmin}_{y \in \boldsymbol{P_h(0,b)}} \mathcal{L}(y) \xrightarrow[b \to \infty]{h \to 0} \operatorname*{argmin}_{y \in \boldsymbol{\mathcal{D}}} \mathcal{L}(y)$$

Constructing approximations

"Limits of $P_h(0,b)$ minimizers are L^2 minimizers"

Theorem

Let $\mathcal{M}_h^b$ be the set of minimizers of $\mathcal{L}$ in $\mathcal{D}_\mathcal{L} \cap P_h(0,b)$. Then any element of

$$\Big\{ f \in L^2(\mathbb{R}^+) \;\Big|\; \exists \text{ two sequences } b_n \to \infty,\ h_n \to 0 \text{ and } \exists \{p_{h_n}^{b_n} \in \mathcal{M}_{h_n}^{b_n}\}_n \text{ s.t. } f = \lim_{n\to\infty} p_{h_n}^{b_n} \Big\}$$

is a minimizer of $\mathcal{L}$ in $\mathcal{D}_\mathcal{L}$

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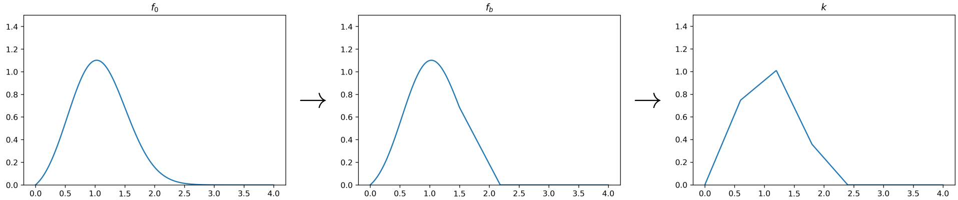
Constructing approximations

Lemma

Let f be a minimizer of $\mathcal{L}$. Then for any $\varepsilon > 0$, there exist $b > 1, h > 0$ and $y \in P_h(0, b)$ such that

$$\|f - y\| < \varepsilon, \qquad |\mathcal{L}(f) - \mathcal{L}(y)| < \varepsilon$$



Phase space uncertainty minimizers

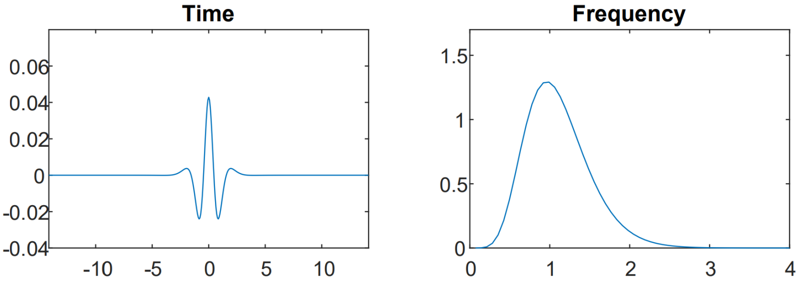


Figure: Numerical spline minimizer of $\mathcal{L}_{\mathrm{P}}$

R. Levie, E. K. Avraham, and N. Sochen, *Wavelet design with optimally localized ambiguity function: a variational approach*, (2021)

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Thank you!

Questions?