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Symbol recovery for localization operators

A quantum harmonic analysis approach

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Time-frequency analysis and localization operators

In time-frequency analysis, a central object is the *short-time Fourier transform* $V_g : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d})$

$$V_g\psi(x,\omega) = \int_{\mathbb{R}^d} \psi(t)\overline{g(t-x)}e^{-2\pi i t\cdot\omega}\,dt.$$

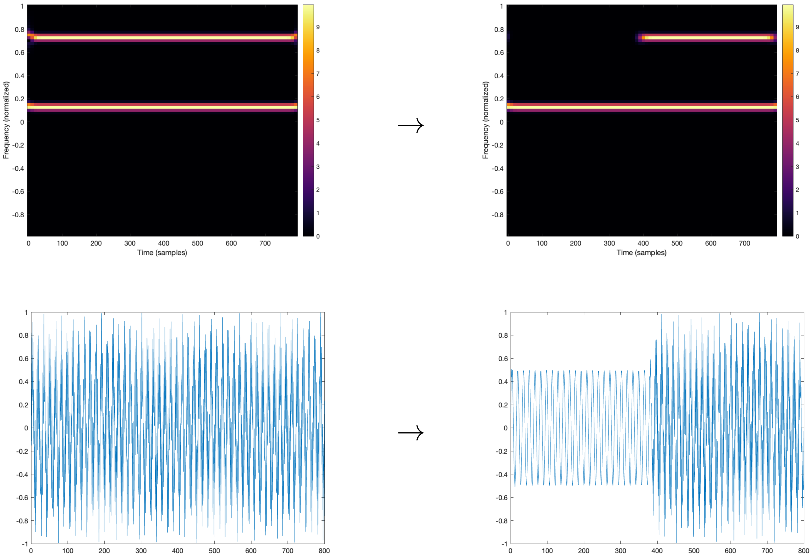
From it, we can recover ψ as

$$\psi = \int_{\mathbb{R}^{2d}} V_g\psi(z)\pi(z)g\,dz.$$

A *localization operator* is constructed by weighing this recovery with a symbol $f : \mathbb{R}^{2d} \to \mathbb{R}$

$$A_f^g\psi = \int_{\mathbb{R}^{2d}} f(z)V_g\psi(z)\pi(z)g\,dz.$$

Examples of localization operator action



The (inverse) problem

Given some information about A_f^g, estimate the symbol f

Previously investigated by:

- Abreu and Dörfler (2012),
- Abreu, Gröchenig and Romero (2014),
- Luef and Skrettingland (2018),
- Romero and Speckbacher (2022)

Four approaches:

- Fourier approach
- Look at spectral data of A_f^g
- Apply A_f^g to white noise
- Tiling the TF plane

Quantum harmonic analysis crash course I

- Function-operator convolutions:

$$f \star S = \int_{\mathbb{R}^{2d}} f(z)\pi(z)S\pi(z)^* \, dz, \qquad f \star (g \otimes g) = A_f^g.$$

- Operator-operator convolutions:

$$T \star S(z) = \operatorname{tr}\left(T\pi(z)S\pi(z)^*\right), \qquad (\psi \otimes \psi) \star (\varphi \otimes \varphi)(z) = |V_\varphi \psi(z)|^2.$$

- Fourier-Wigner transform:

$$\mathcal{F}_W(S) = e^{-\pi i x \cdot \omega} \operatorname{tr}(\pi(-z)S), \qquad \mathcal{F}_W(\varphi \otimes \varphi)(z) = e^{\pi i x \cdot \omega} V_\varphi \varphi(z).$$

Quantum harmonic analysis crash course II

Boundedness:

$$\|f \star S\|_{\mathcal{S}^p} \leq \begin{matrix} \|f\|_{L^1} \|S\|_{\mathcal{S}^p}, \\ \|f\|_{L^p} \|S\|_{\mathcal{S}^1}, \end{matrix}$$

$$\|T \star S\|_{L^p} \leq \|T\|_{\mathcal{S}^p} \|S\|_{\mathcal{S}^1}.$$

Associativity:

$$(f \star S) \star T(z) = f * (S \star T)(z),$$

$$(f * g) \star S = f \star (g \star S).$$

Adjoints:

$$\mathcal{A}_S : L^p(G) \to \mathcal{S}^p, \qquad f \mapsto f \star S,$$

$$\mathcal{B}_S : \mathcal{S}^p \to L^p(G), \qquad T \mapsto T \star S,$$

$$\mathcal{A}_S^* = \mathcal{B}_S.$$

Fourier:

$$\mathcal{F}_W(f \star S) = \mathcal{F}_\sigma(f) \cdot \mathcal{F}_W(S),$$

$$\mathcal{F}_W(T \star S)(z) = \mathcal{F}_W(T)(z) \cdot \mathcal{F}_W(S)(z).$$

Fourier approach

We can try to apply a convolution theorem directly to disentangle the function-operator convolution.

$$\begin{aligned}\mathcal{F}_W(A_f^g)(z) &= \mathcal{F}_W(f \star (g \otimes g))(z) = \mathcal{F}_\sigma(f)(z)\mathcal{F}_W(g \otimes g)(z) \\ &= \mathcal{F}_\sigma(f)(z)A(g)(z) = \mathcal{F}_\sigma\Big(\mathcal{F}_\sigma(f) * W(g)\Big)(z).\end{aligned}$$

This can also be deconvolved!

- Requires full spectral knowledge (to compute $\mathcal{F}_W(A_f^g)$!)
- Requires knowledge of window / blind deconvolution

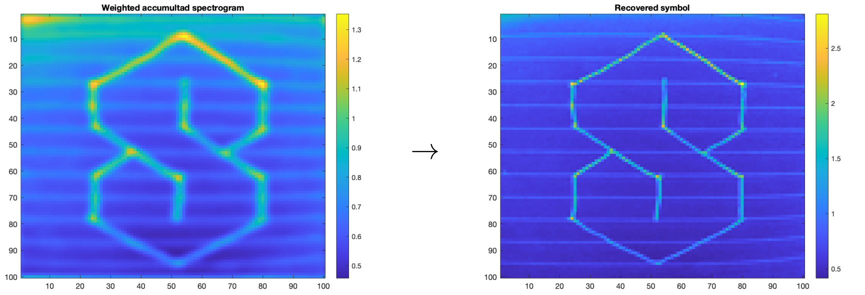
Spectral approach

Convolve with $\varphi \otimes \varphi$:

$$A_f^g \star (\varphi \otimes \varphi)(z) = f * (g \otimes g) \star (\varphi \otimes \varphi)(z) = f * |V_g \varphi|^2(z)$$

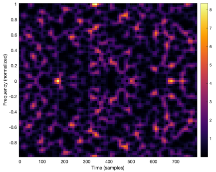
$$= \mathrm{tr}\left(A_f^g \pi(z)(\varphi \otimes \varphi)\pi(z)^*\right) = \left(\sum_k \lambda_k h_k \otimes h_k\right) \star (\varphi \otimes \varphi)(z) = \sum_k \lambda_k |V_\varphi h_k(z)|^2$$

If we know g, this turns into a deconvolution problem:

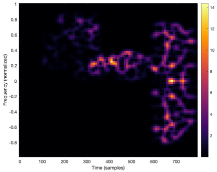


White noise approach

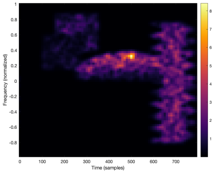
Idea: Spectrogram of white noise is *approximately* uniform



Intuitively: Applying localization operator to white noise should hence weigh this based on f



Improvement: To get rid of noise, take the average over many realizations



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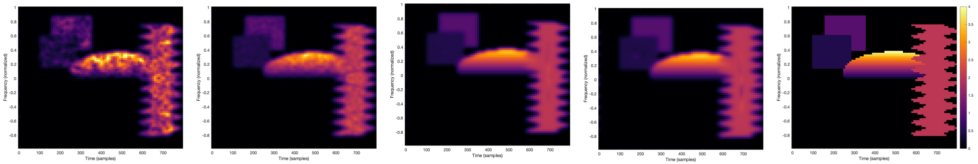
White noise estimator

Formally and visually, what does this look like?

$$\rho(z) = \frac{1}{K}\sum_{k=1}^{K} V_\varphi(A_f^g \mathcal{N}_k) \approx \sum_k \lambda_k^2 |V_\varphi h_k(z)|^2 \approx f^2 * |V_\varphi g|^2(z) \approx f(z)^2.$$

ρ_{20} ρ_{200} ϑ $f^2 * |V_g g|^2$ f^2



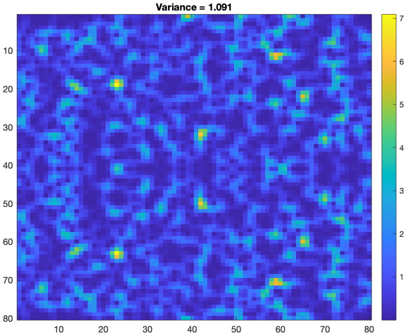
Error estimation:

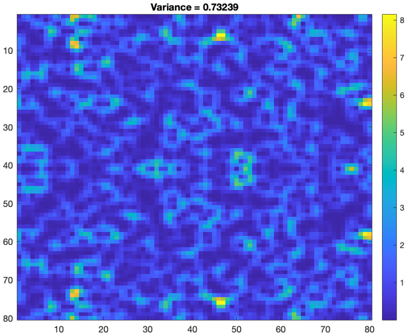
$$\sum_k \lambda_k^2 |V_\varphi h_k(z)|^2 = \left(A_f^g\right)^2 \star (\varphi \otimes \varphi)(z) = \left(A_{f^2}^g + \text{Error}\right) \star (\varphi \otimes \varphi)$$

$$= f^2 * |V_\varphi g|^2 + \text{Error} * (\varphi \otimes \varphi)$$

Optimal white noise

If we have control over the input, we could choose "optimal" white noise with less unevenness.





Plane tiling

Idea: With white noise, we filled the time-frequency plane with white noise - let's instead fill it by an orthonormal basis!

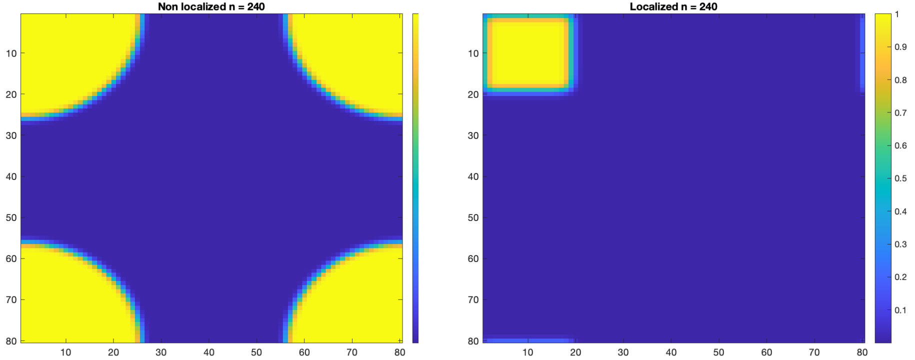
$$\sum_n |V_\varphi e_n(z)|^2 = I \star (\varphi \otimes \varphi)(z) = (1 \star \varphi \otimes \varphi) \star (\varphi \otimes \varphi)(z) = 1 * |V_\varphi \varphi|^2(z) = 1.$$

By replacing $\{e_n\}_n$ by $\{A_f^g e_n\}_n$ this tiling should be weighed by f^2:

$$\sum_n |V_\varphi (A_f^g e_n)(z)|^2 = (f \star (g \otimes g))^2 \star (\varphi \otimes \varphi)(z) \approx f(z)^2.$$

Let's try it out!

Plane tiling example



Frame tiling limitations

This method is expensive if we need many terms! Hence it should only be used when we know something about the support of f, then the inputs can be chosen to cover this area in the time-frequency plane.

We don’t have to use an ONB though - when we have full control over input, it can be replaced by well chosen uniform white noise.

Summary of methods

$$
\begin{aligned}
\mathcal{F}_\sigma(\mathcal{F}_W(f \star (g \otimes g))) &= f * W(g)(z) \approx f(z) \\
\big(f \star (g \otimes g)\big) \star \varphi \otimes \varphi(z) &= f * |V_\varphi g|^2(z) \approx f(z) \\
\begin{matrix} |V_\varphi(A_f^g(\mathcal{N}))|^2 \\ \sum_n |V_\varphi(A_f^g e_n)(z)|^2 \end{matrix} &\approx f^2 * |V_\varphi g|^2(z) \approx f(z)^2
\end{aligned}
$$

Takeaway:
Quantum harmonic analysis provides an appropriate framework to study localization operators!

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Thank you!