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Four ways to recover the symbol of a non-binary localization operator

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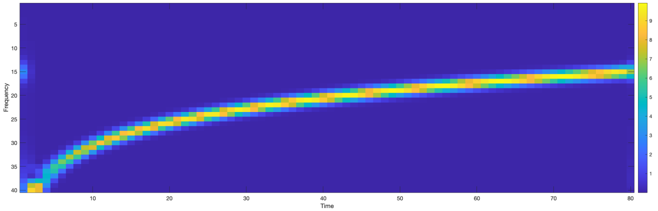
Basics of time-frequency analysis I

In time-frequency analysis, a central object is the *short-time Fourier transform* $V_g : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d})$

$$V_g\psi(x,\omega) = \int_{\mathbb{R}^d} \psi(t)\overline{g(t-x)}e^{-2\pi i t\cdot\omega}\,dt = \langle \psi, \pi(x,\omega)g\rangle.$$

Example: $\psi(t) = \sin(|t|^{1.1})$, then $|V_g\psi|^2$ looks like:



Basics of time-frequency analysis II

From the STFT, we can reconstruct the signal ψ as

$$\psi = \int_{\mathbb{R}^{2d}} V_g\psi(z)\pi(z)g\,dz.$$

A **localization operator** is constructed by weighing this reconstruction with a **symbol** $f : \mathbb{R}^{2d} \to \mathbb{R}$

$$A_f^g\psi = \int_{\mathbb{R}^{2d}} f(z)V_g\psi(z)\pi(z)g\,dz.$$

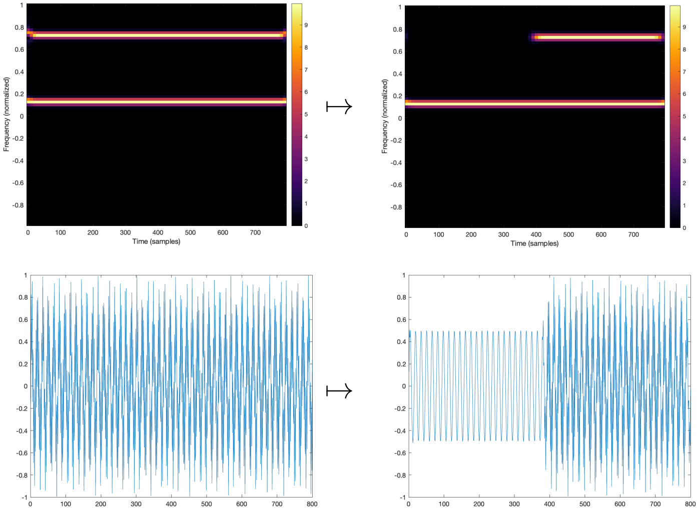
(Non-binary means $f : \mathbb{R}^{2d} \not\to \{0, 1\}$)

Let's look at it visually!

Example of localization operator action

$$\psi(t) = \sin(t) + \sin(5t)$$

$|V_g(\psi)|^2$ $\mapsto$ $|V_g(A_f^g \psi)|^2$



Used for:

- Quantization procedure
- Pseudodifferential operators
- Noise reduction
- Audio effects
- Machine learning preprocessing

The (inverse) problem

Given some information about A_f^g, estimate the symbol f

Previously investigated by:

- Abreu and Dörfler (2012),
- Abreu, Gröchenig and Romero (2014),
- Luef and Skrettingland (2018),
- Romero and Speckbacher (2022)

Four approaches:

- Fourier deconvolution
- Look at spectral data of A_f^g
- Apply A_f^g to white noise
- Tiling the TF plane

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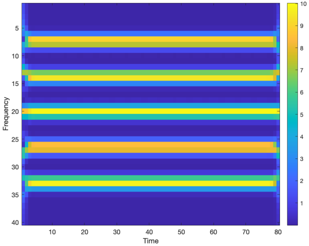
An (informative) naive solution

Reasonable idea:
The earlier $\sin(t) + \sin(5t)$ allowed us to infer some information about the symbol, what if we fill the plane with them?



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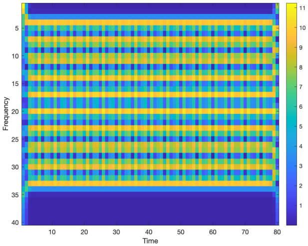

Issue:
Interference, the spectrograms are not additive! Also hard to derive error estimates in this setting - we need some hard analysis.

Quantum harmonic analysis crash course I

- Function-operator convolutions:

$$f \star S = \int_{\mathbb{R}^{2d}} f(z)\pi(z)S\pi(z)^* \, dz, \qquad f \star (g \otimes g) = A_f^g.$$

- Operator-operator convolutions:

$$T \star S(z) = \operatorname{tr}\left(T\pi(z)S\pi(z)^*\right), \qquad (\psi \otimes \psi) \star (\varphi \otimes \varphi)(z) = |V_\varphi \psi(z)|^2.$$

- Fourier-Wigner transform:

$$\mathcal{F}_W(S)(z) = e^{-\pi i x \cdot \omega} \operatorname{tr}(\pi(-z)S), \qquad \begin{aligned} \mathcal{F}_W(\varphi \otimes \varphi)(z) &= e^{\pi i x \cdot \omega} V_\varphi \varphi(z), \\ \mathcal{F}_\sigma(\mathcal{F}_W(\varphi \otimes \varphi))(z) &= W(\varphi). \end{aligned}$$

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Quantum harmonic analysis crash course II

Boundedness:

$$\|f \star S\|_{\mathcal{S}^p} \leq \begin{matrix} \|f\|_{L^1}\|S\|_{\mathcal{S}^p}, \\ \|f\|_{L^p}\|S\|_{\mathcal{S}^1}, \end{matrix}$$

$$\|T \star S\|_{L^p} \leq \|T\|_{\mathcal{S}^p}\|S\|_{\mathcal{S}^1}.$$

Associativity:

$$(f \star S) \star T(z) = f * (S \star T)(z),$$

$$(f * g) \star S = f \star (g \star S).$$

Fourier:

$$\mathcal{F}_W(f \star S)(z) = \mathcal{F}_\sigma(f)(z) \cdot \mathcal{F}_W(S)(z),$$

$$\mathcal{F}_\sigma(T \star S)(z) = \mathcal{F}_W(T)(z) \cdot \mathcal{F}_W(S)(z).$$

Perspective: Symbol recovery = QHA deconvolution

- **Problem**: How to approximately invert $f \mapsto A_f^g = f \star (g \otimes g)$
- **QHA Generalization**: How to approximately invert $f \mapsto f \star S$
- **Uniqueness:**
  $f \mapsto f \star S$ injective $\iff \mathcal{F}_W(S) \neq 0 \iff \mathcal{F}_\sigma(S \star S) \neq 0$

Motivation:
For a *regular* operator S, $L^1(\mathbb{R}^{2d}) \star S$ is dense in $\mathcal{S}^1$. Hence QHA deconvolution is a general dequantization scheme

$$\mathcal{S}^1 \ni A \mapsto f_A \in L^1(\mathbb{R}^{2d}).$$

Can be used to compare operators by comparing associated functions.

Fourier deconvolution

We can apply the convolution theorem to disentangle the function-operator convolution.

$$\mathcal{F}_W(A_f^g)(z) = \mathcal{F}_W[f \star (g \otimes g)](z) = \mathcal{F}_\sigma(f)(z) \cdot \mathcal{F}_W(g \otimes g)(z) = \mathcal{F}_\sigma\Big(f * W(g)\Big)(z)$$

$$\mathcal{F}_\sigma \mathcal{F}_\sigma = I \implies \mathcal{F}_\sigma(\mathcal{F}_W(A_f^g)) = f * W(g)$$

This can be deconvolved!

- Requires full spectral knowledge (to compute $\mathcal{F}_W(A_f^g)$!)
- Requires knowledge of window / blind deconvolution

(We have computed the Weyl symbol of A_f^g)

Weighted accumulated Wigner estimator

If $S = g \otimes g$, then

$$f \star S = \sum_k \lambda_k (h_k \otimes h_k) \implies f * W(g) = \sum_k \lambda_k W(h_k)$$

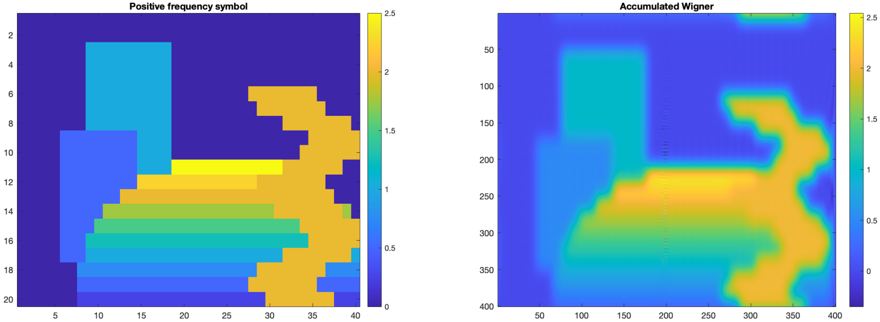


Figure: A symbol and the corresponding weighted accumulated Wigner estimator.

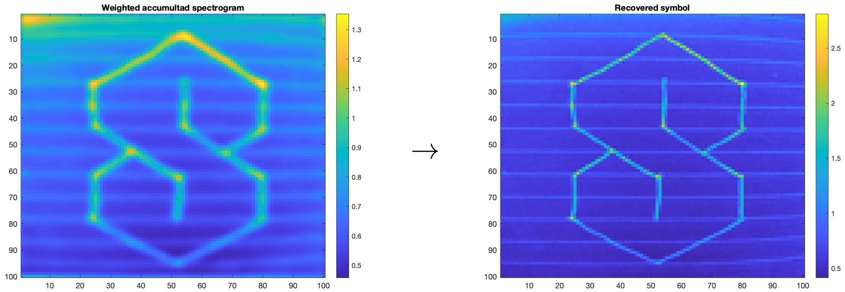
Spectral approach

Convolve with $\varphi \otimes \varphi$:

$$A_f^g \star (\varphi \otimes \varphi)(z) = f * (g \otimes g) \star (\varphi \otimes \varphi)(z) = f * |V_g \varphi|^2(z)$$

$$= \left(\sum_k \lambda_k (h_k \otimes h_k) \right) \star (\varphi \otimes \varphi)(z) = \sum_k \lambda_k |V_\varphi h_k(z)|^2$$

If we know g, this turns into a deconvolution problem:



Spectral approach - pure operator formulation

Taking the viewpoint of inverting $f \mapsto f \star S$, we can make this approach a bit clearer:

$$(f \star S) \star S = f * (S \star S).$$

If we don't know S, we can make a guess:

$$(f \star S) \star T = f * (S \star T).$$

Want to choose T so that $S \star T$ is well-concentrated.

Spectral approach - result formulation

Theorem

Let $f \in L^1(\mathbb{R}^{2d})$ be real-valued and of bounded variation and $g \in L^2(\mathbb{R}^d)$ with $\|g\|_{L^2} = 1$. Then if $A_f^g = \sum_k \lambda_k (h_k \otimes h_k)$,

$$\left\| \sum_{k=1}^{N} \lambda_k |V_g h_k|^2 - f \right\|_{L^1} \leq \sum_{k=N+1}^{\infty} |\lambda_k| + \mathrm{Var}(f) \int_{\mathbb{R}^{2d}} |z| |V_g g(z)|^2 \, dz.$$

Moreover, in the $N = \infty$ case,

$$\sum_{k=1}^{\infty} \lambda_k |V_g h_k(z)|^2 = f * |V_g g|$$

which can be deconvolved in the sense that

$$f = \mathcal{F}_\sigma^{-1} \left(\frac{\mathcal{F}_\sigma(f * (S \star S))}{\mathcal{F}_\sigma(S \star S)} \right).$$

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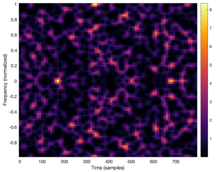
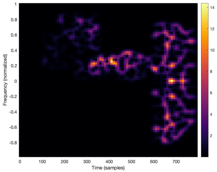
White noise approach

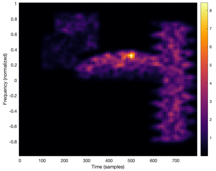
Idea: Spectrogram of white noise is *approximately* uniform



Intuitively: Applying localization operator to white noise should hence weigh this based on f



Improvement: To get rid of noise, take the average over many realizations



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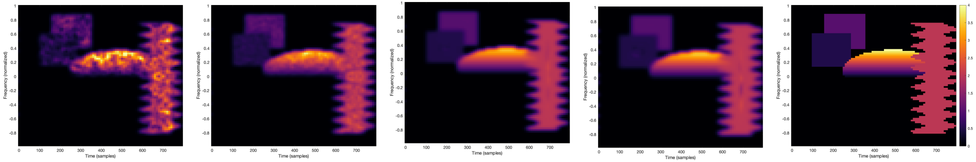
White noise estimator

Formally and visually, what does this look like?

$$\rho(z) = \frac{1}{K}\sum_{k=1}^{K} |V_\varphi(A_f^g \mathcal{N}_k)(z)|^2 \approx \sum_m \lambda_m^2 |V_\varphi h_m(z)|^2 \approx f^2 * |V_\varphi g|^2(z) \approx f(z)^2.$$

ρ_{20} ρ_{200} ϑ $f^2 * |V_g g|^2$ f^2



Error estimation:

$$\sum_m \lambda_m^2 |V_\varphi h_m(z)|^2 = \left(A_f^g\right)^2 \star (\varphi \otimes \varphi)(z) = \left(A_{f^2}^g + \text{Error}\right) \star (\varphi \otimes \varphi)$$

$$= f^2 * |V_\varphi g|^2 + \text{Error} * (\varphi \otimes \varphi)$$

White noise L^1 error

Theorem

Let $f \in C_c^{d+2}(\mathbb{R}^{2d})$, ρ be given by $\rho(z) = \frac{1}{K}\sum_{k=1}^{K} |V_\varphi(A_f^g \mathcal{N}_k)(z)|^2$ with white noise variance σ^2, $g, \varphi \in \mathcal{S}(\mathbb{R}^d)$ with $\|g\|_{L^2} = \|\varphi\|_{L^2} = 1$. Then there exists a constant B dependent on f, g, φ and a constant $C < 1$ such that

$$\mathbb{P}\left(\int_{\mathbb{R}^{2d}} \left|\frac{\rho(z)}{\sigma^2} - f(z)^2\right| dz > B + t\right) \leq \frac{\|f\|_{L^2}^2}{t\sqrt{K}} \left[\frac{3}{2\sqrt{\pi C}} \operatorname{erf}\left(\sqrt{CK}\right) + \frac{3}{C\sqrt{K}} e^{-CK}\right]$$
$$= O\left(\frac{1}{\sqrt{K}}\right).$$

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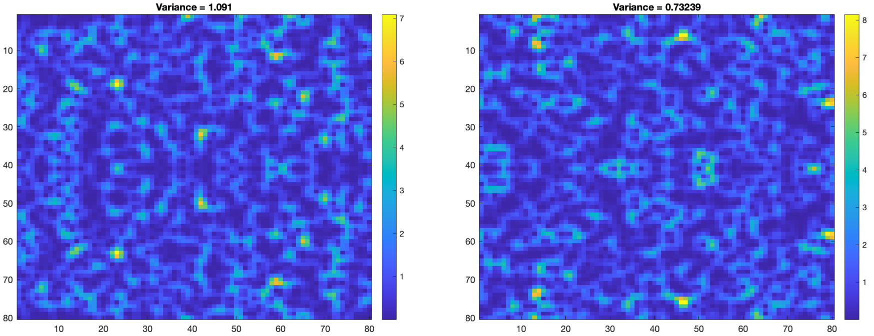
"Optimal" white noise

If we have control over the input, we could choose "optimal" white noise with less unevenness.



Drawback is we lose probabilistic tools.

Plane tiling

Idea: With white noise, we filled the time-frequency plane with white noise - let's instead fill it by an orthonormal basis!

$$\sum_n |V_\varphi e_n(z)|^2 = I \star (\varphi \otimes \varphi)(z) = (1 \star \varphi \otimes \varphi) \star (\varphi \otimes \varphi)(z) = 1 * |V_\varphi \varphi|^2(z) = 1.$$

By replacing $\{e_n\}_n$ by $\left\{A_f^g e_n\right\}_n$ this tiling will be weighed by f^2:

$$\begin{aligned}
\sum_n |V_\varphi(A_f^g e_n)(z)|^2 &= \sum_n (A_f^g e_n \otimes A_f^g e_n) \star (\varphi \otimes \varphi)(z) \\
&= A_f^g \left(\sum_n e_n \otimes e_n \right) A_f^g \star (\varphi \otimes \varphi)(z) \\
&= (A_f^g I A_f^g) \star (\varphi \otimes \varphi)(z) \\
&= (A_f^g)^2 \star (\varphi \otimes \varphi)(z) \approx A_{f^2}^g \star (\varphi \otimes \varphi)(z) \approx f(z)^2.
\end{aligned}$$

Let's try it out!

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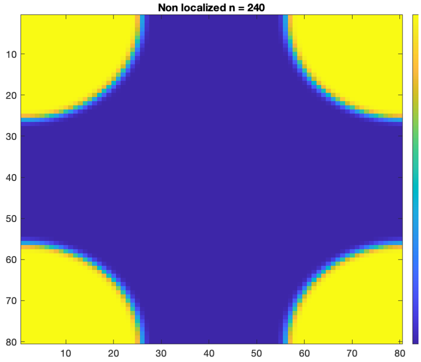
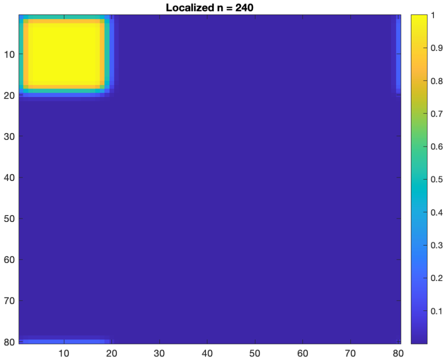
Plane tiling example

$$\sum_{n=0}^{0306090120150180210240} |V_\varphi e_n(z)|^2 \qquad \sum_{n=0}^{0306090120150180210240} |V_\varphi (A_f^g e_n)(z)|^2$$





Summary of methods (rank one)

$$\begin{aligned}
\mathcal{F}_\sigma(\mathcal{F}_W(f \star (g \otimes g))) &= f * W(g)(z) \approx f(z) \\
\big(f \star (g \otimes g)\big) \star \varphi \otimes \varphi(z) &= f * |V_\varphi g|^2(z) \approx f(z) \\
\frac{|V_\varphi(A_f^g(\mathcal{N}))|^2}{\sum_n |V_\varphi(A_f^g e_n)(z)|^2} &\approx f^2 * |V_\varphi g|^2(z) \approx f(z)^2
\end{aligned}$$

Future work

Mixed-state localization operators ($f \star S$):

- Requires $\frac{1}{K}\sum_{k=1}^{K} Q_S(f \star S(\mathcal{N}_k)) \xrightarrow{k\to\infty} \sum_m \lambda_m^2 Q_S(h_m)$

Replacing white noise by more general "ambient" input

- If $|V_\varphi(\mathcal{N})|^2 \approx \rho$, then $|V_\varphi(A_f^g \mathcal{N})|^2 \approx f^2 \cdot \rho$?

Replace white noise by "optimal" input

- If we can find $U \in L^2(\mathbb{R}^d)$ such that $|V_\varphi(U)|^2 \approx 1$ on some large ball, then $|V_\varphi(A_f^g U)|^2$ should approximate f^2 on that ball.

Optimal T in $f \approx f * (S \star T)$

- In the rank-one case, given g find h that minimizes

$$\int_{\mathbb{R}^{2d}} |z| |V_g h(z)|^2 \, dz.$$

Thank you!