NTNU

Norwegian University of Science and Technology

Extensions of Quantum Harmonic Analysis and Applications to Time-Frequency Analysis

PhD Defense

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June 11, 2025

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Opponents: Elena Cordero and Bruno Torresani

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Extensions of

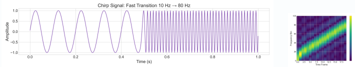
Quantum harmonic analysis

$$\alpha_z(S) = \pi(z) S \pi(z)^*$$

- Operator convolutions
- Operator Fourier transform
- Weyl quantization

... and applications to

Time-frequency analysis



- Short-time Fourier transform
- Localization operators
- Discretization

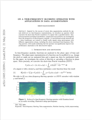
dealt with in papers:


A


B


C


D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F


G

First, some basics

Outline

Time-frequency analysis

Quantum harmonic analysis

Papers

Outline

Problem: Fourier transform is insufficient

Our starting point is the **Fourier transform**

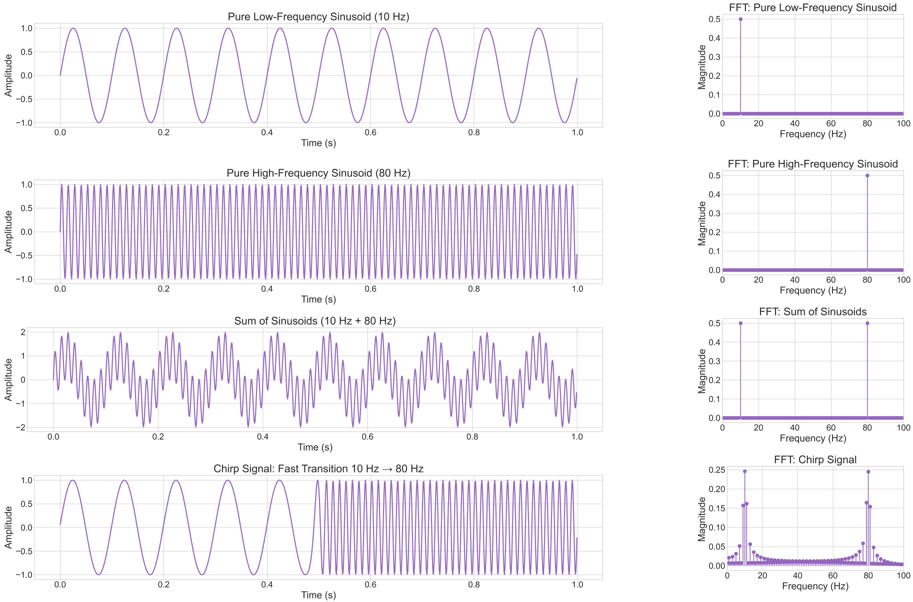
$$\mathcal{F}(f)(\omega) = \hat{f}(\omega) = \int_{\mathbb{R}} f(t) e^{-2\pi i t\omega}\, dt.$$

We care about signals where frequency varies over time, but

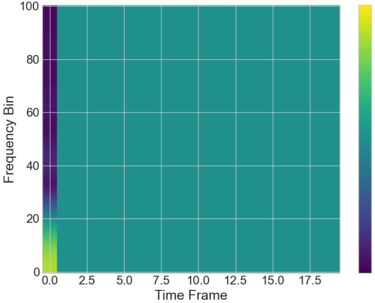
$$\mathcal{F}(f(\cdot - x))(\omega) = e^{2\pi i x\omega} \mathcal{F}(f)(\omega) \implies |\hat{f}(\omega)| = |\widehat{T_x f}(\omega)|,$$

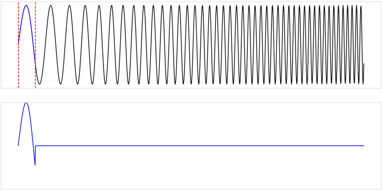
i.e., the ***spectrum*** $|\hat{f}|$ is invariant under translations T_x.

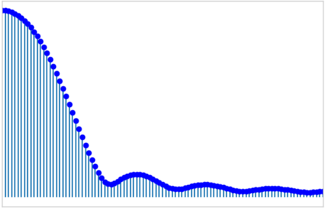
An illustrative example



Solution: Joint time-frequency representation







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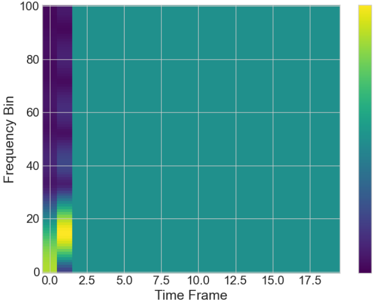

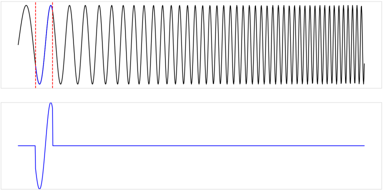

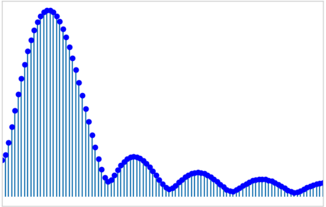

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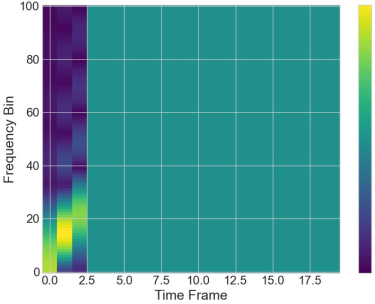

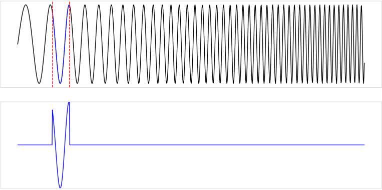

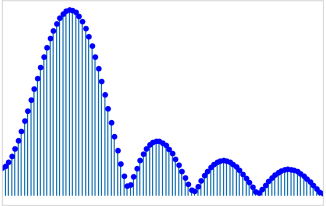

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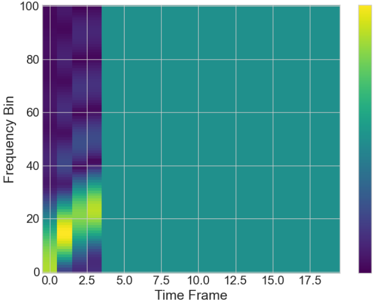

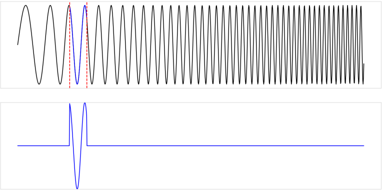

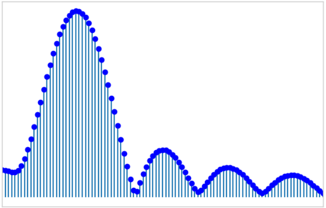

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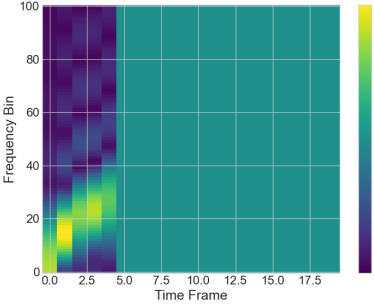

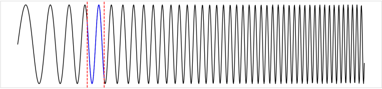

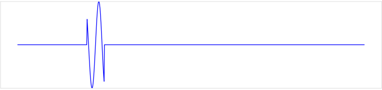

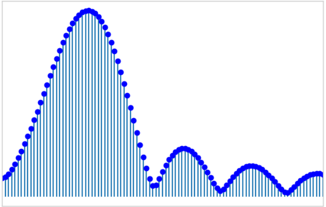

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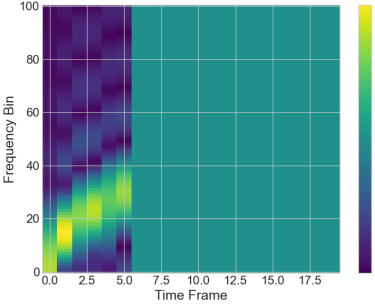

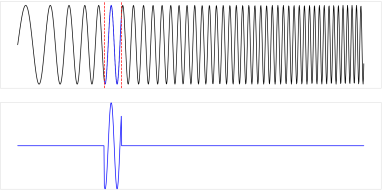

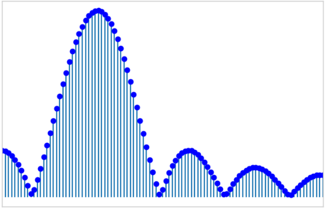

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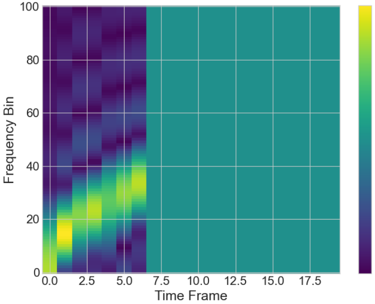

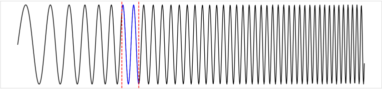

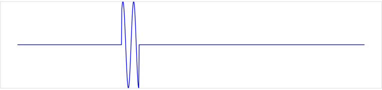

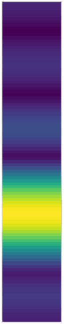

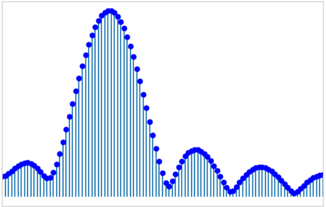

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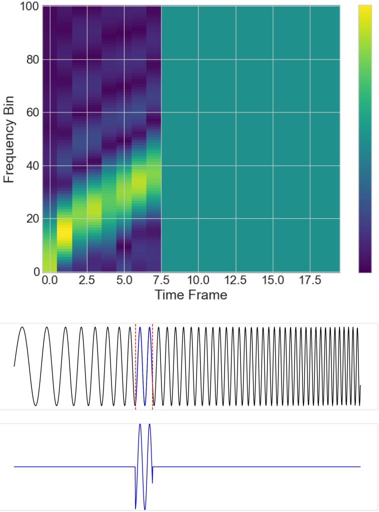

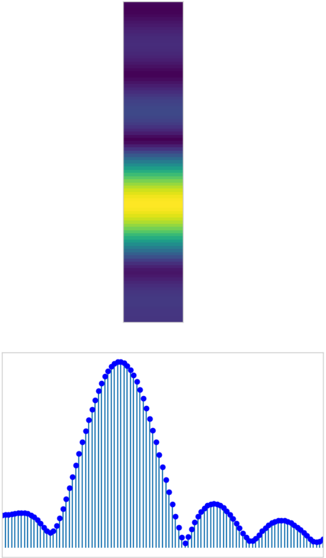

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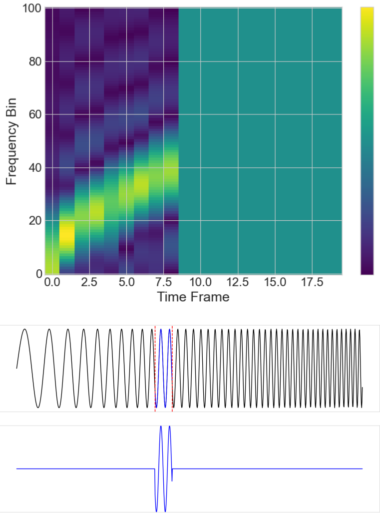

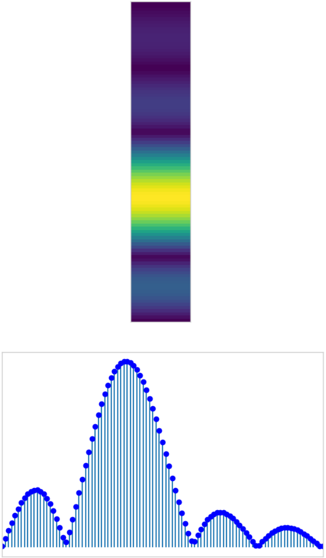

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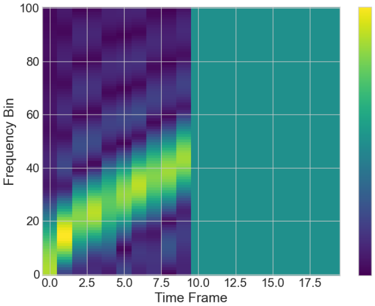

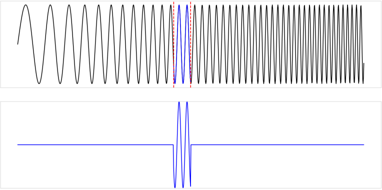

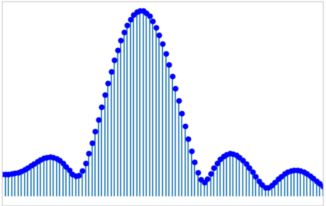

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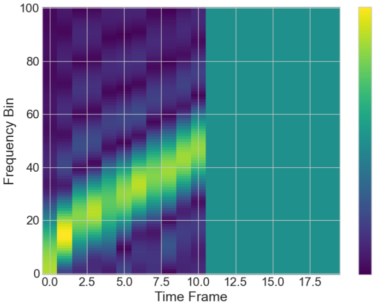

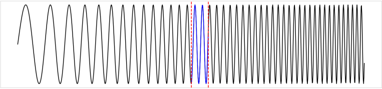

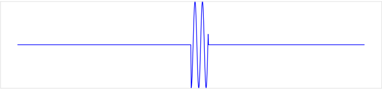

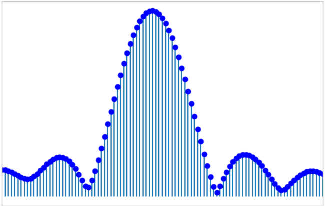

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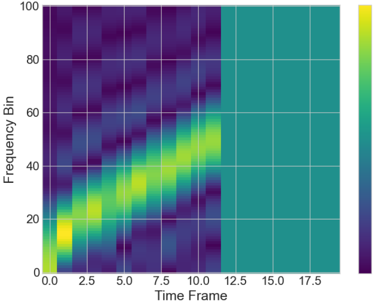

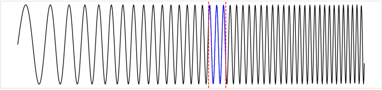

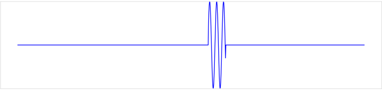

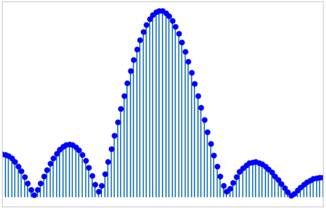

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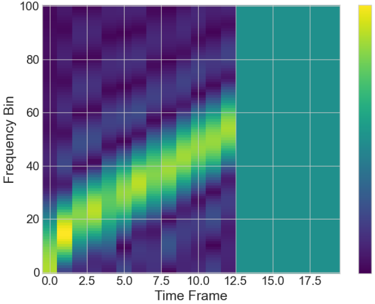

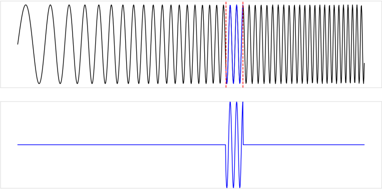

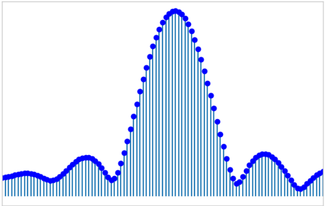

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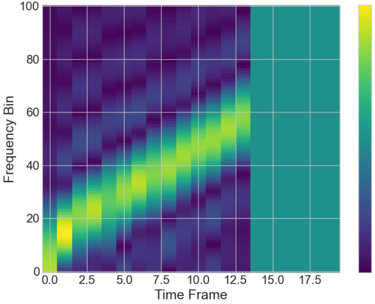

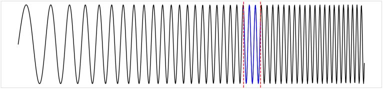

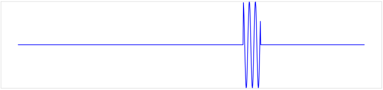

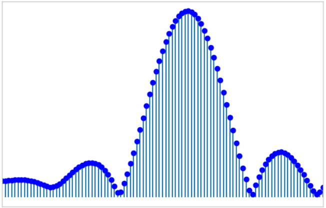

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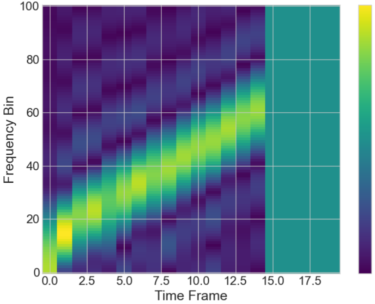

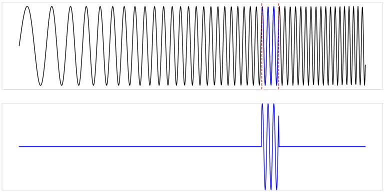

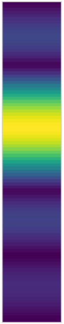

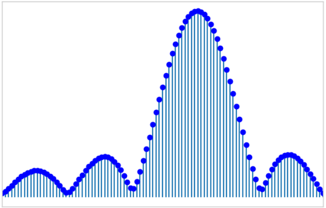

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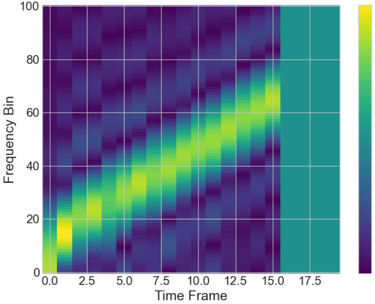

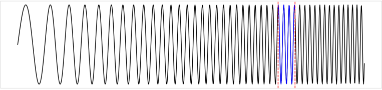

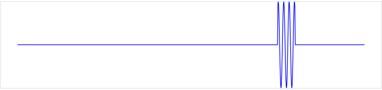

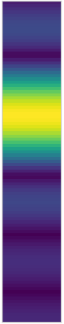

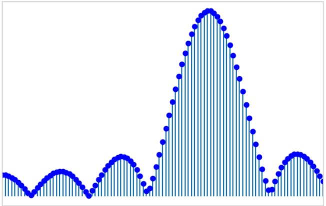

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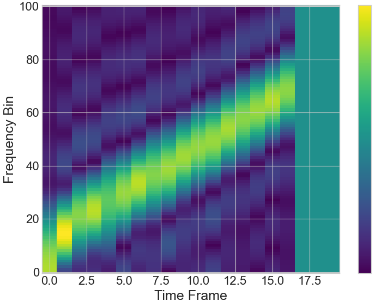

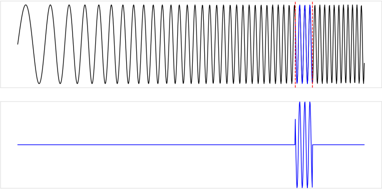

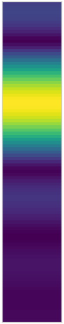

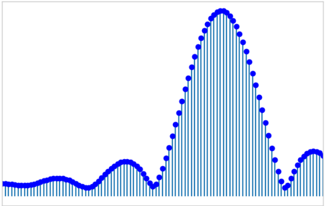

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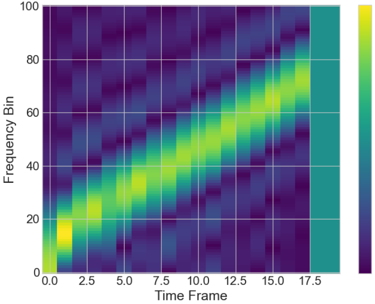

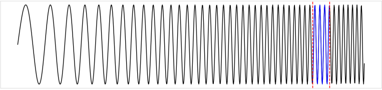

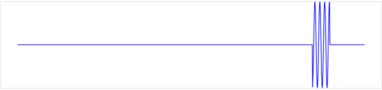

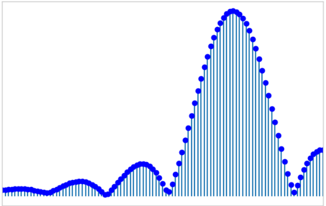

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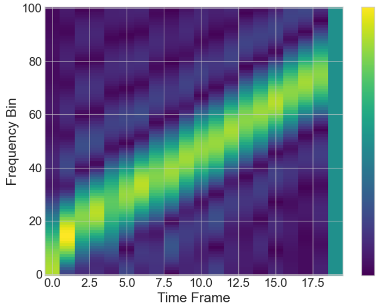

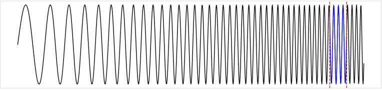

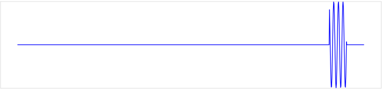

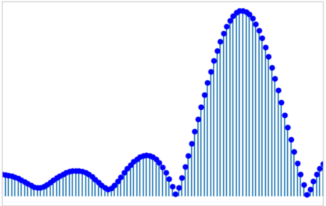

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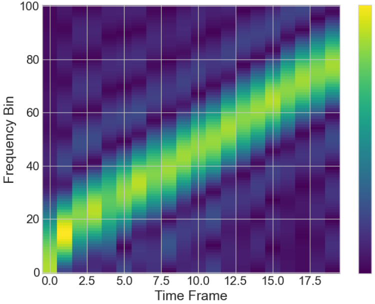
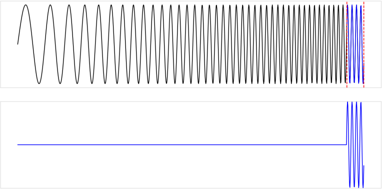
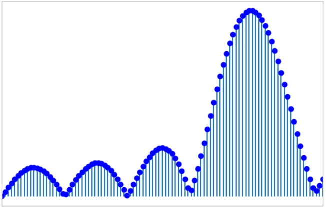






Detailing the STFT

We call this the **short-time Fourier transform** (STFT)

$$V_g f(x,\omega) = \mathcal{F}(f(\cdot)\overline{g(\cdot - x)})(\omega) = \int_{\mathbb{R}} f(t)\overline{g(t-x)}e^{-2\pi i t\omega}\,dt.$$

$V_g : L^2(\mathbb{R}) \to L^2(\mathbb{R}^2)$ is a linear **time-frequency representation** and

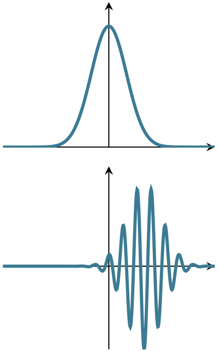
$$\langle V_g f_1, V_g f_2 \rangle_{L^2(\mathbb{R}^2)} = \langle f_1, f_2 \rangle_{L^2(\mathbb{R})}$$

when g is chosen appropriately.

Using

- Translation $T_x f(t) = f(t-x)$
- Modulation $M_\omega f(t) = e^{2\pi i \omega t} f(t)$
- Time-frequency shift $\pi(x,\omega)f = M_\omega T_x f$

we can write $V_g f(x,\omega) = \langle f, \pi(x,\omega)g\rangle_{L^2}$. We will write $z = (x,\omega) \in \mathbb{R}^2$ as a shorthand.



Reconstruction

The adjoint of the STFT mapping, $V_g^* : L^2(\mathbb{R}^2) \to L^2(\mathbb{R})$

$$V_g^* = \int_{\mathbb{R}^2} F(z)\pi(z)g\, dz,$$

is a right inverse of the STFT, but not a left inverse:

$$\underbrace{V_g^* V_g = I_{L^2(\mathbb{R})},}_{} \qquad \underbrace{V_g V_g^* = P_{V_g(L^2(\mathbb{R}))}}_{\text{orthogonal projection}}.$$

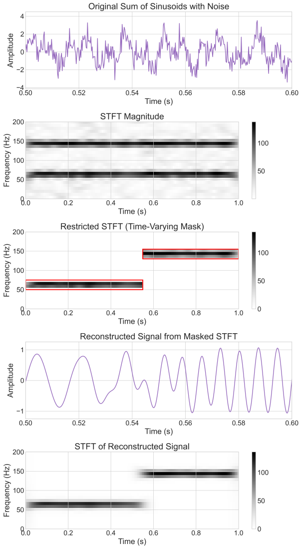
$$\implies \boxed{\boldsymbol{f = \int_{\mathbb{R}^2} V_g f(z)\pi(z)g\, dz}}$$

<u>Not every $F \in L^2(\mathbb{R}^2)$ can be written as $F = V_g f$ for some $f \in L^2(\mathbb{R})$</u>

Restricting the reconstruction

By multiplying $V_g f$ by a function $m : \mathbb{R}^2 \to \mathbb{C}$ prior to reconstruction, we get a **localization operator:**

$$A_m^g f = \int_{\mathbb{R}^2} m(z) V_g f(z) \pi(z) g \, dz$$

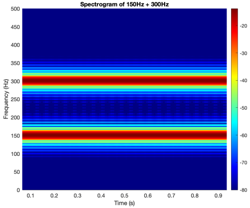


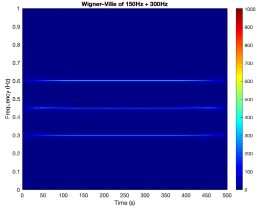
Time-frequency distributions

- Spectrogram, squared modulus of STFT $|V_g f|^2$
- Wigner distribution

$$W(f,g)(x,\omega) = \int_{\mathbb{R}} f(t - x/2)\overline{g(t + x/2)}e^{-2\pi i \omega t}\, dt$$

- Smoothed versions of the Wigner distributions (Cohen's class)





Gabor frames / discretization

We can only sample $V_g f$ at discrete points

- $\int_{\mathbb{R}^2} |V_g f(z)|^2 \, dz \; = \; \|f\|_{L^2}^2$ (continuous)
- $\sum_{\lambda \in \Lambda} |V_g f(\lambda)|^2 \; \sim \; \|f\|_{L^2}^2$ (discrete)

We say $\Lambda \subset \mathbb{R}^2$ induces a **Gabor frame** if

$$A\|f\|_{L^2}^2 \; \leq \; \sum_{\lambda \in \Lambda} |V_g f(\lambda)|^2 \; \leq \; B\|f\|_{L^2}^2,$$

for some $A, B > 0$.

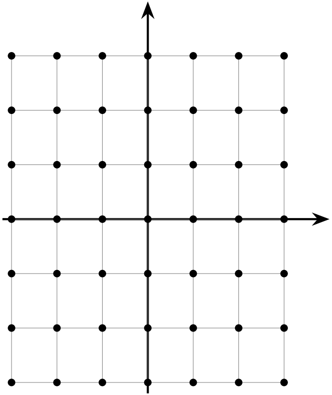


Figure: Example of a subset Λ of $\mathbb{R}^2$ which can be used for sampling.

Outline

Generalizing harmonic analysis to operators

In harmonic analysis we deal with functions f and their:

- Translations T_x
- Integrals $\int$
- Fourier transform $\mathcal{F}$
- Convolutions $*$
- L^p spaces

We want to set up similar notions for operators $S : L^2(\mathbb{R}) \to L^2(\mathbb{R})$:

- Translations $\alpha_z(S) = \pi(z)S\pi(z)^*$
- Traces $\mathrm{tr}(S) = \sum_n \langle Se_n, e_n \rangle$
- Fourier transform $\mathcal{F}_W$
- Convolutions $\star$
- $\|S\|_{\mathcal{S}^p} = \mathrm{tr}(|S|^p)^{1/p}$

Operator convolutions

QHA meta statement: Replace

- Functions $\to$ Operators $\quad g \to S$
- Translations $\to$ Operator translations $\quad T_z \to \alpha_z$
- Integrals $\to$ Traces $\quad \int \to \mathrm{tr}$

$$f * g(z) = \int_{\mathbb{R}^{2d}} f(y) T_z \check{g}(y)\, dy$$

$$f \star S = \int_{\mathbb{R}^{2d}} f(z) \alpha_z(S)\, dz$$

(Function-operator convolution)

$$S \star T(z) = \mathrm{tr}(S \alpha_z(\check{T}))$$

(Operator-operator convolution)

An operator Fourier transform

There is already a well-known Fourier transform for operators, the **Fourier-Wigner** transform

$$\mathcal{F}_W : \mathcal{S}^1 \to C_0(\mathbb{R}^2), \qquad \mathcal{F}_W(S)(z) = \mathrm{tr}(S\pi(-z)).$$

Riemann–Lebesgue

For functions on $\mathbb{R}^{2d}$ we will use the **symplectic** Fourier transform

$$\mathcal{F}_\sigma(f)(z) = \int_{\mathbb{R}^2} f(z')\, e^{-2\pi i\, \sigma(z,z')}\, dz'.$$

Standard properties hold

Harmonic analysis	Quantum harmonic analysis
$\\|f * g\\|_{L^p} \leq \\|f\\|_{L^1}\\|g\\|_{L^p}$	$\\|f \star S\\|_{\mathcal{S}^p} \leq \\|f\\|_{L^1}\\|S\\|_{\mathcal{S}^p}$ $\\|T \star S\\|_{L^p} \leq \\|T\\|_{\mathcal{S}^1}\\|S\\|_{\mathcal{S}^p}$
$(f * g) * h = f * (g * h)$	$f * (S \star T) = (f \star S) \star T$ $f \star (g \star T) = (f * g) \star T$
$\mathcal{F}(f * g) = \mathcal{F}(f) \cdot \mathcal{F}(g)$	$\mathcal{F}_W(f \star S) = \mathcal{F}_\sigma(f) \cdot \mathcal{F}_W(S)$ $\mathcal{F}_\sigma(T \star S) = \mathcal{F}_W(T) \cdot \mathcal{F}_W(S)$

Weyl quantization

Weyl quantization is a map from functions on phase space to operators on $L^2(\mathbb{R})$, $f \mapsto A_f$

$$\begin{tikzcd}
\big(\mathcal{S}^2(L^2(\mathbb{R})),\, \circ,\, {}^*\big) \arrow[d, "\mathcal{F}_W"] \arrow[dr, dotted, "a"] & \\
\big(L^2(\mathbb{R}^2),\, \natural,\, \overline{\check{\ }}\big) \arrow[r, "\mathcal{F}_\sigma"] & \big(L^2(\mathbb{R}^2),\, \sharp,\, \overline{\ }\big)
\end{tikzcd}$$

It is an isometric bijection from $L^2(\mathbb{R}^2)$ to $\mathcal{S}^2(L^2(\mathbb{R}))$.

Operator convolutions = convolutions of Weyl symbols:

$$A_{f*g} = f \star A_g, \qquad A_f \star A_g = f * \check{g}.$$

Outline

Paper A: Quantum harmonic analysis on locally compact groups

Published in Journal of Functional Analysis

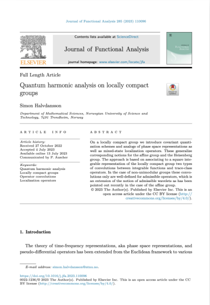
Like in abstract harmonic analysis, we replace

- $\mathbb{R}^{2d} \longrightarrow \boxed{G}$
- $L^2(\mathbb{R}^d) \longrightarrow \boxed{\mathcal{H}}$
- $\pi : \mathbb{R}^{2d} \to \mathcal{U}(L^2(\mathbb{R}^d)) \longrightarrow \boxed{\sigma : G \to \mathcal{U}(\mathcal{H})}$

Functions $f \in L^1(G)$, operators $S \in \mathcal{S}^1(\mathcal{H})$

In abstract time-frequency analysis, we deal with **admissibility** of wavelets. For us, $T \star S \in L^1(G)$ is dependent on

$$\mathcal{D}^{-1} S \mathcal{D}^{-1} \in \mathcal{S}^1 \qquad \iff \qquad S \text{ is an admissible operator.}$$



Journal of Functional Analysis 285 (2023) 110096

Contents lists available at ScienceDirect

Journal of Functional Analysis

journal homepage: www.elsevier.com/locate/jfa

Full Length Article

Quantum harmonic analysis on locally compact groups

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ARTICLE INFO

Article history:
Received 27 October 2022
Accepted 4 July 2023
Available online 13 July 2023
Communicated by P. Auscher

Keywords:
Quantum harmonic analysis
Locally compact groups
Operator convolutions
Localization operators

ABSTRACT

On a locally compact group we introduce covariant quantization schemes and analogs of phase space representations as well as mixed-state localization operators. These generalize corresponding notions for the affine group and the Heisenberg group. The approach is based on associating to a square integrable representation of the locally compact group two types of convolutions between integrable functions and trace-class operators. In the case of non-unimodular groups these convolutions only are well-defined for admissible operators, which is an extension of the notion of admissible wavelets as has been pointed out recently in the case of the affine group.

1. Introduction

The theory of time-frequency representations, aka phase space representations, and pseudo-differential operators has been extended from the Euclidean framework to various

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https://doi.org/10.1016/j.jfa.2023.110096

Paper B: Measure-operator convolutions and applications to mixed-state Gabor multipliers

Published in Sampling Theory, Signal Processing, and Data Analysis, joint work with Franz Luef and Hans Feichtinger

Function-operator convolution:

$$f \star S = \int_{\mathbb{R}^{2d}} f(z)\alpha_z(S)\,dz.$$

Perhaps **measure**-operator convolution is

$$\mu \star S = \int_{\mathbb{R}^{2d}} \alpha_z(S)\,d\mu(z)?$$

Goals:

- Motivate definition from first principles
- Use results to study Gabor multipliers which can be realized as measure-operator convolutions


Sampling Theory, Signal Processing, and Data Analysis (2024) 22:16
https://doi.org/10.1007/s43670-024-00090-0
ORIGINAL ARTICLE

Measure-operator convolutions and applications to mixed-state Gabor multipliers

Hans G. Feichtinger[1,2] · Simon Halvdansson[3] · Franz Luef[3]

Received: 1 September 2023 / Accepted: 6 June 2024 / Published online: 3 July 2024

Abstract
For the Weyl-Heisenberg group, convolutions between functions and operators were defined by Werner as a part of a framework called quantum harmonic analysis. We show how recent results by Feichtinger can be used to extend this definition to include convolutions between measures and operators. Many properties of function-operator convolutions carry over to this setting and allow us to prove novel results on the distribution of eigenvalues of mixed-state Gabor multipliers and derive a version of the Berezin-Lieb inequality for lattices. New results on the continuity of Gabor multipliers with respect to lattice parameters, masks and windows as well as their ability to approximate localization operators are also derived using this framework.

Keywords Quantum harmonic analysis · Operator convolutions · Homogeneous Banach spaces · Gabor multipliers · Gabor frames

1 Introduction

In recent years the framework of quantum harmonic analysis, which was introduced by R. Werner in [36], has been successfully applied to many problems in time-frequency analysis and operator theory [18, 26, 28, 29]. The most central operations of the framework are the *function-operator* and *operator-operator* convolutions which generalize

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Birkhäuser

Extending actions

- Standard convolutions can be defined by extending the action $\mathbb{R} \times L^1(\mathbb{R}) \ni (x, f) \mapsto T_x f$ to $M(\mathbb{R}) \times L^1(\mathbb{R})$
- We do the same for $\mathbb{R}^2 \times \mathcal{S}^1 \ni (z, S) \mapsto \alpha_z(S)$ to get a form of weighted translation

Theorem

The map $\bullet_\rho : \mathbb{R}^{2d} \times \mathcal{S}^1 \to \mathcal{S}^1, (z, S) \mapsto \pi(z)S\pi(z)^*$ *has a unique bounded essential extension to* $M(\mathbb{R}^{2d}) \times \mathcal{S}^1 \to \mathcal{S}^1$. *That extension satisfies*

$$\langle (\mu \star S)f, g \rangle = \int \langle \pi(z)S\pi(z)^* f, g \rangle \, d\mu(z).$$

NTNU

Application: Approximating localization operators

The Gabor multiplier $G^g_{m,\alpha,\beta}$ associated to the lattice $\Lambda_{\alpha,\beta} = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$ can be written as

$$G^g_{m,\alpha,\beta} = \mu^m_{\alpha,\beta} \star (g \otimes g)$$

where $\mu^m_{\alpha,\beta}$ is a discrete measure.

Theorem
Let $(\mu_\alpha)_\alpha$ be a bounded and tight net which converges weak- to μ_0 and $S \in \mathcal{S}^1$. Then*

$$\lim_{\alpha\to\infty} \left\|\mu_\alpha \star S - \mu_0 \star S\right\|_{\mathcal{S}^1} = 0.$$

Theorem
Let $m \in W(L^\infty, \ell^1)(\mathbb{R}^{2d})$ be Riemann-integrable and $S \in \mathcal{S}^1$. Then

$$\lim_{\alpha,\beta\to 0} \left\|\mu^m_{\alpha,\beta} \star S - m \star S\right\|_{\mathcal{S}^1} = 0.$$

In particular, $\|G^g_{m,\alpha,\beta} - A^g_m\|_{\mathcal{S}^1} \to 0$ as $\alpha, \beta \to 0$.

Paper C: Weyl Quantization of Exponential Lie Groups for Square Integrable Representations

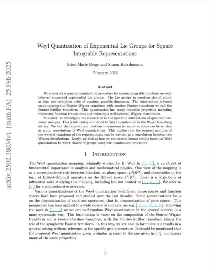
Preprint, joint work with Stine Marie Berge

Goal:

Set up quantization beyond Weyl-Heisenberg and affine groups.

- Need connected exponential Lie group and square integrable representation
- Replace symplectic Fourier transform by *Fourier-Kirillov* transform

$$
\begin{tikzcd}
\left(\mathcal{S}^2(\mathcal{H}),\, \circ,\, {}^*\right) \arrow[d, "\mathcal{F}_W"] \arrow[dr, dotted, "a"] & \\
\left(\mathcal{F}_W(\mathcal{S}^2),\, \natural,\, \sqrt{\Delta(\cdot)}\,\check{}\,\right) \arrow[r, "\mathcal{F}_{KO}"] & \left(L^2_r(G),\, \sharp,\, {}^-\right)
\end{tikzcd}
$$



Weyl Quantization of Exponential Lie Groups for Square Integrable Representations

Stine Marie Berge and Simon Halvdansson

February 2025

arXiv:2502.18034v1 [math.FA] 25 Feb 2025

Abstract

We construct a general quantization procedure for square integrable functions on well-behaved connected exponential Lie groups. The Lie groups in question should admit at least one co-adjoint orbit of maximal possible dimension. The construction is based on composing the Fourier-Wigner transform with another Fourier transform we call the Fourier-Kirillov transform. This quantization has many desirable properties including respecting function translations and inducing a well-behaved Wigner distribution.

Moreover, we investigate the connection to the operator convolutions of quantum harmonic analysis. This is intricately connected to Weyl quantization in the Weyl-Heisenberg setting. We find that convolution relations in quantum harmonic analysis can be written as group convolutions of Weyl quantizations. This implies that the squared modulus of the wavelet transform of the representation can be written as a convolution between two Wigner distributions. Lastly, we look at how we can extend known results based on Weyl quantization to wider classes of groups using our quantization procedure.

1 Introduction

The Weyl quantization mapping, originally studied by H. Weyl in [51, 52], is an object of fundamental importance in analysis and mathematical physics. One view of the mapping is as a correspondence rule between functions on phase space, $L^2(\mathbb{R}^{2d})$, and observables in the form of Hilbert-Schmidt operators on the Hilbert space $L^2(\mathbb{R}^d)$. There is a large body of influential work studying this mapping, including but not limited to [47, 45, 14]. We refer to [54] for a comprehensive overview.

Various generalizations of the Weyl quantization to different phase spaces and function spaces have been proposed and studied over the last decades. Some generalizations focus on the dequantization of rank-one operators, that is, dequantization of pure states. This perspective has been applied to a wide variety of contexts, see e.g. [13, 44, 28, 3, 46]. Following the work in [24, 10] we set out to formulate Weyl quantization in the general context in a more systematic way. This formulation is based on the composition of the Fourier-Wigner transform and a *Fourier-Kirillov transform*, with the Fourier-Kirillov transform taking the role of the symplectic Fourier transform. In this way, we are able to formulate our results in a general setting without reference to the specific group structure. It should be mentioned that the proposed Weyl quantization given is similar in spirit to the one given in [43], and enjoys many of the same properties.

1

Quantization properties

- **Translation and conjugation** are respected

$$\alpha_x(A_f) = A_{f(\cdot x^{-1})}, \qquad A_f^* = A_{\bar{f}}.$$

- **The map** is a unitary H^*-algebra isomorphism

$$A : L_r^2(G) \to \mathcal{S}^2(\mathcal{H}).$$

- **Wigner distribution** can be realized as dequantization of rank-one operator

$$W(\psi, \phi)(x) = a_{\psi\otimes\phi}(x) = \mathcal{F}_{\mathsf{KO}}(\mathcal{F}_{\mathsf{W}}(\psi \otimes \phi))(x),$$

not the object whici induces the quantization.

Paper D: Five ways to recover the symbol of a non-binary localization operator

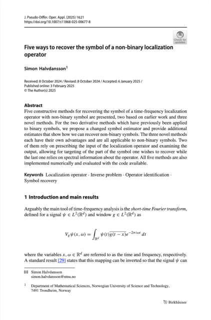
Published in Journal of Pseudo-Differential Operators and Applications

Standard problem: Find Ω from information about A_Ω^g

- Previously studied by Abreu, Dörfler, Gröchenig, Romero, Luef, Skrettingland, Speckbacher
- Used eigenfunctions and image of white noise

Goal:

- Adapt old methods to work for A_m^g where $m \in L^1(\mathbb{R}^2)$
- Develop new methods
- Implement all methods in MATLAB



J. Pseudo-Differ. Oper. Appl. (2025) 16:21
https://doi.org/10.1007/s11868-025-00677-8

Five ways to recover the symbol of a non-binary localization operator

Simon Halvdansson[1]

Received: 8 October 2024 / Revised: 8 October 2024 / Accepted: 6 January 2025 / Published online: 3 February 2025

Abstract
Five constructive methods for recovering the symbol of a time-frequency localization operator with non-binary symbol are presented, two based on earlier work and three novel methods. For the two derivative methods which have previously been applied to binary symbols, we propose a changed symbol estimator and provide additional estimates that show how we can recover non-binary symbols. The three novel methods each have their own advantages and are all applicable to non-binary symbols. Two of them rely on prescribing the input of the localization operator and examining the output, allowing for targeting of the part of the symbol one wishes to recover while the last one relies on spectral information about the operator. All five methods are also implemented numerically and evaluated with the code available.

Keywords Localization operator · Inverse problem · Operator identification · Symbol recovery

1 Introduction and main results

Arguably the main tool of time-frequency analysis is the *short-time Fourier transform*, defined for a signal $\psi \in L^2(\mathbb{R}^d)$ and window $g \in L^2(\mathbb{R}^d)$ as

$$V_g\psi(x,\omega) = \int_{\mathbb{R}^d} \psi(t)\overline{g(t-x)}e^{-2\pi i \omega t}\,dt$$

where the variables $x, \omega \in \mathbb{R}^d$ are referred to as the time and frequency, respectively. A standard result [29] states that this mapping can be inverted so that the signal ψ can

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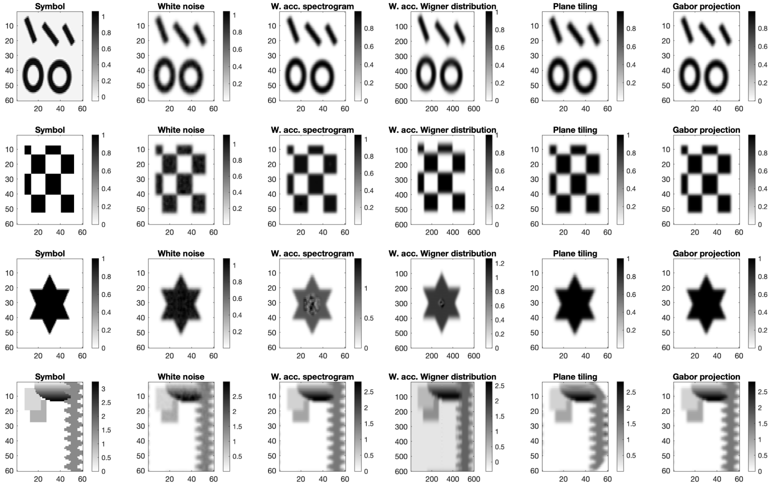
Birkhäuser

Formulations

$$A_m^g = \sum_k \lambda_k (h_k \otimes h_k)$$

- $\sum_k \lambda_k |V_g h_k(z)|^2 \leftarrow$ Weighted accumulated spectrogram
- $\sum_k \lambda_k W(h_k)(z) \leftarrow$ Weighted accumulated Wigner distribution
- $\frac{1}{K} \sum_{k=1}^{K} |V_g(A_m^g \mathcal{N})(z)|^2 \leftarrow$ White noise estimator
- $\sum_n |V_g(A_m^g e_n)(z)|^2 \leftarrow$ Plane tiling estimator
- $V_g(A_m^g(\pi(z)g))(z) \leftarrow$ Gabor projection

Examples



Symbol
White noise
W. acc. spectrogram
W. acc. Wigner distribution
Plane tiling
Gabor projection

Paper E: On a time-frequency blurring operator with applications in data augmentation

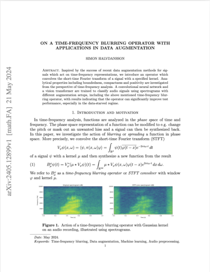
Published in Journal of Fourier Analysis and Applications

What if instead of multiplying the STFT (localization operator) we convolve it (blurring operator)?

$$B^g_\mu f = V_g^*(\mu * V_g f)$$

Mathematically, we look at:

- Boundedness of operator between L^p, M^p and Schwartz spaces
- (Non)-compactness
- Positivity condition



arXiv:2405.12899v1 [math.FA] 21 May 2024

ON A TIME-FREQUENCY BLURRING OPERATOR WITH APPLICATIONS IN DATA AUGMENTATION

SIMON HALVDANSSON

Abstract. Inspired by the success of recent data augmentation methods for signals which act on time-frequency representations, we introduce an operator which convolves the short-time Fourier transform of a signal with a specified kernel. Analytical properties including boundedness, compactness and positivity are investigated from the perspective of time-frequency analysis. A convolutional neural network and a vision transformer are trained to classify audio signals using spectrograms with different augmentation setups, including the above mentioned time-frequency blurring operator, with results indicating that the operator can significantly improve test performance, especially in the data-starved regime.

1. Introduction and motivation

In time-frequency analysis, functions are analyzed in the *phase space* of time and frequency. The phase space representation of a function can be modified to e.g. change the pitch or mask out an unwanted hiss and a signal can then be synthesized back. In this paper, we investigate the action of *blurring* or *spreading* a function in phase space. More precisely, we convolve the short-time Fourier transform (STFT)

$$V_\varphi\psi(x,\omega) = \langle \psi, \pi(x,\omega)\varphi\rangle = \int_{\mathbb{R}^d} \psi(t)\overline{\varphi(t-x)}e^{-2\pi i\omega\cdot t}\,dt$$

of a signal ψ with a kernel μ and then synthesize a new function from the result

$$B^\varphi_\mu\psi(t) = V_\varphi^*(\mu * V_\varphi\psi)(t) = \int_{\mathbb{R}^{2d}} \mu * V_\varphi\psi(x,\omega)\varphi(t-x)e^{2\pi i\omega\cdot t}\,dx\,d\omega. \quad (1)$$

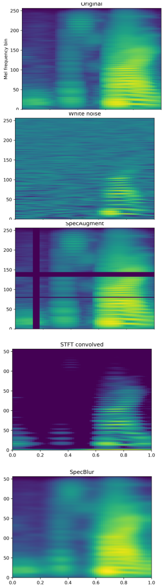
We refer to B^φ_μ as a *time-frequency blurring operator* or *STFT convolver* with window φ and kernel μ.

Figure 1. Action of a time-frequency blurring operator with Gaussian kernel on an audio recording, illustrated using spectrograms.

Date: May 2024.
Keywords: Time-frequency blurring, Data augmentation, Machine learning, Audio preprocessing.

1

Application



The operator shows promise as a **data augmentation** tool.

Table: Average ViT test accuracies with standard errors (%) for different augmentation setups.

Augmentation	**Accuracy**
None	$89.17_{\pm 0.20}$
White noise	$90.72_{\pm 0.09}$
SpecAugment	$90.61_{\pm 0.14}$
STFT-blur	$90.40_{\pm 0.15}$
SpecBlur	$91.29_{\pm 0.13}$
White noise + SpecAug	$91.80_{\pm 0.15}$
STFT-blur + SpecBlur	$91.72_{\pm 0.12}$
All	$92.70_{\pm 0.08}$

Paper F: On accumulated spectrograms for Gabor frames

Published in Journal of Mathematical Analysis and Applications

Classical result:

If $A_\Omega^g = \sum_k \lambda_k (h_k \otimes h_k)$, then

$$\left\| \sum_{k=1}^{\lceil |\Omega| \rceil} |V_g h_k|^2 - \chi_\Omega \right\|_{L^1} \leq C_g |\partial\Omega|.$$

accumulated spectrogram

Goal:

Show corresponding results for the **Gabor multiplier** $G_{\Omega,\Lambda}^g$ associated to the lattice Λ.


J. Math. Anal. Appl. 543 (2025) 129044

Contents lists available at ScienceDirect

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Regular Articles

On accumulated spectrograms for Gabor frames

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ARTICLE INFO

Article history:
Received 3 May 2024
Available online 12 November 2024
Submitted by S. Tikhonov

Keywords:
Accumulated spectrogram
Gabor multiplier
Localization operator
Gabor frame
Weyl-Heisenberg ensemble

ABSTRACT

Analogs of classical results on accumulated spectrograms, the sum of spectrograms of eigenfunctions of localization operators, are established for Gabor multipliers. We show that the lattice ℓ^1 distance between the accumulated spectrogram and the indicator function of the Gabor multiplier mask is bounded by the number of lattice points near the boundary of the mask and that this bound is sharp in general. The methods developed for the proofs are also used to show that the Weyl-Heisenberg ensemble restricted to a lattice is hyperuniform when the Gabor frame is tight.

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1. Introduction and main results

In time-frequency analysis, localization operators restrict a signal f to a subset Ω of the time-frequency plane [7] by means of a restricted resolution of the identity via the short-time Fourier transform (STFT) $V_g f$ as

$$A_\Omega^g f = \int_\Omega V_g f(z) \pi(z) g \, dz, \qquad V_g f(z) = \langle f, \pi(z) g \rangle$$

where $g \in L^2(\mathbb{R}^d)$ is a *window function*, $z = (x, \omega) \in \mathbb{R}^{2d}$ is a point in *time-frequency space* and $\pi(z)f(t) = \pi(x,\omega)f(t) = e^{2\pi i \omega \cdot t} f(t - x)$ is a *time-frequency shift* [17]. The spectral behavior of such operators has been studied extensively [2,11,13,21,26,28,29], showing that there are approximately $|\Omega|$ eigenvalues close to 1, followed by a *plunge region* of size comparable to the length of the perimeter of Ω, after which the remaining eigenvalues are close to 0. In [3], Abreu, Gröchenig and Romero showed that Ω can be estimated from only the spectrograms of the first $\lceil |\Omega| \rceil$ eigenfunctions using the *accumulated spectrogram*

$$\rho_\Omega(z) = \sum_{k=1}^{\lceil |\Omega| \rceil} |V_g h_k^\Omega(z)|^2 \qquad (1)$$

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https://doi.org/10.1016/j.jmaa.2024.129044

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Results

We only observe $\Omega \cap \Lambda$, consequently

- Errors are in $\ell^1(\Lambda)$ instead of $L^1(\mathbb{R}^{2d})$
- We measure the perimeter by $\partial^r_\Lambda \Omega = \Lambda \cap (\partial\Omega + B(0,r))$.

Theorem

Let $g \in M^*_\Lambda(\mathbb{R}^d)$ and Λ be such that (g, Λ) induces a frame with frame constants $A, B > 0$, $r > 0$ and $\Omega \subset \mathbb{R}^{2d}$ be compact. Then there exists a constant C depending only on r and d such that

$$\|\rho_\Omega - \chi_\Omega\|_{\ell^1(\Lambda)} \leq C_g \# \partial^{r_\Lambda}_\Lambda \Omega + 2\frac{B-A}{B} \#(\Omega \cap \Lambda) + \frac{B}{\|g\|^2_{L^2}}$$

where $r_\Lambda = r + l_M$ and l_M is the diameter of the fundamental domain of Λ.

- $\boldsymbol{A = B \implies \|\rho_\Omega - \chi_\Omega\|_{\ell^1(\Lambda)} \leq C_g \# \partial^{r_\Lambda}_\Lambda + D}$
- Estimate is tight:

$$C_1 \# \partial^{r_\Lambda}_\Lambda B(0,R) \leq \|\rho_{B(0,R)} - \chi_{B(0,R)}\|_{\ell^1(\Lambda)} \leq C_2 \# \partial^{r_\Lambda}_\Lambda B(0,R)$$

Paper G: Empirical plunge profiles of time-frequency localization operators

Preprint

For localization operators A_Ω^g:

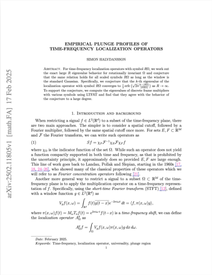
- First $\sim |\Omega|$ eigenvalues ≈ 1
- Then $\lesssim |\partial\Omega|$ eigenvalues not 1 nor 0 (plunge region)
- Remaining eigenvalues ≈ 0

Lots of related results but no progress since late 1980s.

Goal: Describe eigenvalue behavior in more detail

Approach:

- Only eigenvalues for $\Omega = B(0, R)$ known
- Extend this result to more Ω
- Conjecture universality
- Test numerically



arXiv:2502.11805v1 [math.FA] 17 Feb 2025

EMPIRICAL PLUNGE PROFILES OF TIME-FREQUENCY LOCALIZATION OPERATORS

SIMON HALVDANSSON

Abstract. For time-frequency localization operators with symbol $R\Omega$, we work out the exact large R eigenvalue behavior for rotationally invariant Ω and conjecture that the same relation holds for all scaled symbols $R\Omega$ as long as the window is the standard Gaussian. Specifically, we conjecture that the k-th eigenvalue of the localization operator with symbol $R\Omega$ converges to $\frac{1}{2}\operatorname{erfc}\left(\sqrt{2\pi}\frac{k-|R\Omega|}{|\partial R\Omega|}\right)$ as $R \to \infty$. To support the conjecture, we compute the eigenvalues of discrete frame multipliers with various symbols using LTFAT and find that they agree with the behavior of the conjecture to a large degree.

1. Introduction and background

When restricting a signal $f \in L^2(\mathbb{R}^d)$ to a subset of the time-frequency plane, there are two main approaches. The simpler is to consider a spatial cutoff, followed by a Fourier multiplier, followed by the same spatial cutoff once more. For sets $E, F \subset \mathbb{R}^{2d}$ and $\mathcal{F}$ the Fourier transform, we can write such operators as

$$Sf = \chi_F \mathcal{F}^{-1} \chi_E \mathcal{F} \chi_F f \tag{1}$$

where χ_Ω is the indicator function of the set Ω. While such an operator does not yield a function compactly supported in both time and frequency, as that is prohibited by the uncertainty principle, it approximately does so provided E, F are large enough. This line of work goes back to Landau, Pollak and Slepian, starting in the 1960s [17, 18, 24–26], who showed many of the classical properties of these operators which we will refer to as *Fourier concentration operators* following [21].

Another more general way to restrict a signal to a subset $\Omega \subset \mathbb{R}^{2d}$ of the time-frequency plane is to apply the multiplication operator on a time-frequency representation of f. Specifically, using the *short-time Fourier transform* (STFT) [12], defined with a window function $g \in L^2(\mathbb{R}^d)$ as

$$V_g f(x, \omega) = \int_{\mathbb{R}^d} f(t)\overline{g(t-x)} e^{-2\pi i \omega t}\, dt = \langle f, \pi(x,\omega) g\rangle,$$

where $\pi(x,\omega)f(t) = M_\omega T_x f(t) = e^{2\pi i \omega t} f(t-x)$ is a *time-frequency shift*, we can define the *localization operator* A_Ω^g as

$$A_\Omega^g f = \int_\Omega V_g f(x,\omega)\pi(x,\omega) g\, dx\, d\omega.$$

Date: February 2025.
Keywords: Time-frequency, localization operator, universality, plunge region

1

Rotationally-invariant symbol + conjecture

Theorem

Let $\Omega \subset \mathbb{R}^2$ be a ***compact, regular closed and rotationally invariant set with a finite number of connected components****, and let $\lambda_k^{R\Omega}$ the k-th eigenvalue of $A_\Omega^{g_0}$. Then*

$$\left|\lambda_k^{R\Omega} - \frac{1}{2}\operatorname{erfc}\left(\sqrt{2\pi}\frac{k-|R\Omega|}{|\partial R\Omega|}\right)\right| = O\left(\frac{1}{R}\right).$$

Conjecture

Let $\Omega \subset \mathbb{R}^2$ be ***compact and regular closed****, and let λ_k^{Ω} be the k-th eigenvalue of $A_\Omega^{g_0}$. Then*

$$\left|\lambda_k^{R\Omega} - \frac{1}{2}\operatorname{erfc}\left(\sqrt{2\pi}\frac{k-|R\Omega|}{|\partial R\Omega|}\right)\right| = O\left(\frac{1}{R}\right).$$

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Numerics

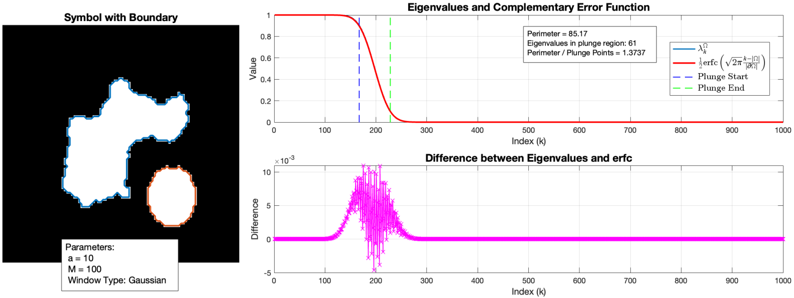

Figure: Symbol, eigenvalues and discrepancy to erfc

Thank you!