Hardy's uncertainty principle - From a historical overview to new developments

Trial Lecture

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Uncertainty principle

Meta-theorem:
A function f and its Fourier transform $\hat{f}$ cannot both be well-localized.

Theorem (Paley-Wiener)
Let $f \in L^2(\mathbb{R})$. If both f and $\hat{f}$ have compact support, then $f = 0$.

Remark:
We will use the normalization

$$\hat{f}(\omega) = \int_{\mathbb{R}} f(x) e^{-2\pi i \omega x} \, dx.$$

Theorem (Heisenberg, Kennard, Weyl, Robertson)

Let $f \in L^2(\mathbb{R})$, then

$$\left(\int_{\mathbb{R}} (x-a)^2 |f(x)|^2 \, dx\right)\left(\int_{\mathbb{R}} (\omega-b)^2 |\hat{f}(\omega)|^2 \, d\omega\right) \geq \frac{\|f\|_2^4}{16\pi^2}.$$

Theorem (Donoho-Stark)

Let $f \in L^2(\mathbb{R})$ with $\|f\|_{L^2} = 1$, then

$$\|f\|_{L^2(\mathbb{R}\setminus E)} < \varepsilon_E, \quad \|\hat{f}\|_{L^2(\mathbb{R}\setminus F)} < \varepsilon_F \implies |E| \cdot |F| \geq \big(1 - (\varepsilon_E + \varepsilon_F)\big)^2.$$

Theorem (Hirschman–Beckner)

Let $f \in L^2(\mathbb{R})$ and define $H(f) = -\int_{\mathbb{R}} |f(x)|^2 \log |f(x)|^2 \, dx$, then

$$H(f) + H(\hat{f}) \geq \log \frac{e}{2}.$$

Full statement

Theorem (Hardy '33)

Let $f \in L^2(\mathbb{R})$ *satisfy* $|f(x)| \leq C_1 e^{-\pi a x^2}$ *and* $|\hat{f}(\omega)| \leq C_2 e^{-\pi b \omega^2}$*. Then*

- $ab > 1 \implies f(x) = 0,$
- $ab = 1 \implies f(x) = C e^{-a\pi x^2}.$

To help keep track of things, we will remember facts in blue boxes and use them via green boxes.

We'll start with $ab = 1$

Proof (1/12): Simplifications

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Theorem (Hardy '33)

Let $f \in L^2(\mathbb{R})$ *satisfy* $|f(x)| \leq C_1 e^{-\pi a x^2}$ *and* $|\hat{f}(\omega)| \leq C_2 e^{-\pi b \omega^2}$. *Then*

- $ab > 1 \implies f(x) = 0,$
- $ab = 1 \implies f(x) = Ce^{-a\pi x^2}.$

- Replace $f \mapsto f/\max\{C_1, C_2\} \implies C_1, C_2 \leq 1$ WLOG.
- Assume true for $a = 1$. For f as in theorem, consider

$$g(x) = f(x/\sqrt{a}) \implies \hat{g}(\omega) = \sqrt{a}\hat{f}(\sqrt{a}\omega)$$
$$\implies |g(x)| \leq e^{-\pi x^2}, \quad |\hat{g}(\omega)| \leq \sqrt{a} e^{-\pi a b \omega^2}.$$

Applying theorem to g, we get that $g(x) = Ce^{-\pi x^2}$ if $ab = 1$ and in particular $f(x) = Ce^{-\pi a x^2}$. WLOG $a = 1$ from now on.

$|f(x)| \leq e^{-\pi x^2}$ $|\hat{f}(\omega)| \leq e^{-\pi \omega^2}$

Proof (2/12): Analytic extension of $\hat{f}$

Define

$$F(z) = \int_{\mathbb{R}} f(u) e^{-2\pi i z u}\, du \qquad \boxed{|\hat{f}(\omega)| \le e^{-\pi\omega^2}} \quad \boxed{|F(x)| \le e^{-\pi x^2}}$$

Claim: F is entire.
Proof: For any fixed $u \in \mathbb{R}$, $f(u)e^{-2\pi i z u}$ is analytic in z. In general,

$$|e^z| = |e^{\mathrm{Re}(z)+i\,\mathrm{Im}(z)}| = |e^{\mathrm{Re}(z)}| \cdot |e^{i\,\mathrm{Im}(z)}| = e^{\mathrm{Re}(z)} \qquad \boxed{|e^z| = e^{\mathrm{Re}(z)}}$$

In particular, using $\boxed{|f(x)| \le e^{-\pi x^2}}$,

$$|f(u)| \cdot |e^{-2\pi i z u}| \le e^{-\pi u^2} e^{2\pi u\,\mathrm{Im}(z)}$$

which is integrable in u. So for each fixed z, the integrand in $F(z)$ is integrable and entire and consequently $\boxed{F \text{ is entire}}$.

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Proof (3/12): Bounding $|F|$

We will need to bound $|F(z)|$. Using $|f(x)| \leq e^{-\pi x^2}$

$$|F(x+iy)| \leq \int_{\mathbb{R}} |f(u)||e^{-2\pi i(x+iy)u}|\,du \leq \int_{\mathbb{R}} e^{-\pi u^2+2\pi yu}\,du.$$

Completing the square,

$$-\pi u^2 + 2\pi yu = -\pi(u^2 - 2yu + y^2) + \pi y^2 = -\pi(u-y)^2 + \pi y^2,$$

meaning that

$$|F(x+iy)| \leq \int_{\mathbb{R}} e^{-\pi u^2+2\pi yu}\,du = e^{\pi y^2} \underbrace{\int_{\mathbb{R}} e^{-\pi(u-y)^2}\,du}_{=1}$$

$|F(x+iy)| \leq e^{\pi y^2}$ $\quad$ $|F(iy)| \leq e^{\pi y^2}$

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Proof (4/12): Defining and bounding G along axes

Forming the function $G(z) = e^{\pi z^2} F(z)$, $|F(x)| \leq e^{-\pi x^2}$ and $|F(iy)| \leq e^{\pi y^2}$ yields

$$|G(x)| \leq e^{\pi x^2} e^{-\pi x^2} = 1 \qquad |G(x)| \leq 1$$

$$|G(iy)| \leq e^{\pi (iy)^2} e^{\pi y^2} = 1 \qquad |G(iy)| \leq 1$$

Since F is entire, it follows that G is entire.

We want to show that G is bounded so that Liouville allows us to conclude that

$$G \equiv C \implies F(z) = Ce^{-\pi z^2} \implies \hat{f}(\omega) = Ce^{-\pi \omega^2} \implies f(x) = Ce^{-\pi x^2}.$$

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Proof (5/12): Phragmén–Lindelöf setup

Theorem (Phragmén–Lindelöf principle)

Let $S = \{z \in \mathbb{C} : \alpha < \arg z < \beta\}$ be a sector of $\mathbb{C}$ and F a function which is analytic in S that is continuous on $\overline{S}$. If $|F(z)| \leq M$ on ∂S and

$$|F(z)| \leq Ce^{c|z|^\rho}$$

for some $c, C > 0$ and $\rho < \frac{\pi}{\beta - \alpha}$, then $|F(z)| \leq M$ on $\overline{S}$.

Using $|F(x+iy)| \leq e^{\pi y^2}$ and $|e^z| = e^{\mathrm{Re}(z)}$,

$$|G(z)| = |G(x+iy)| = |e^{\pi(x+iy)^2}||F(x+iy)| \leq |e^{\pi(x^2+2ixy-y^2)}|e^{\pi y^2} = e^{\pi x^2} \leq e^{\pi|z|^2}$$

$|G(x+iy)| \leq e^{\pi x^2}$

$|G(z)| \leq e^{\pi|z|^2}$

so we have $\rho = 2$ growth.
With $\alpha = 0, \beta = \pi/2$, we need $\rho < \frac{\pi}{\pi/2 - 0} = 2$ to apply Phragmén–Lindelöf.

Proof (6/12): Defining H_δ

For small $\delta > 0$, consider

$$S_\theta = \{z \in S : 0 < \arg z < \theta\}$$

where $\theta = \theta(\delta)$ is dependent on δ.

Define $H_\delta(z) = e^{i\delta z^2} G(z)$, then $|G(x)| \leq 1$ gives

$$|H_\delta(x)| = |e^{i\delta x^2}| \cdot |G(x)| \leq 1 \qquad |H_\delta(x)| \leq 1$$

Along the θ ray parametrized by $z = re^{i\theta}$, using $|G(x+iy)| \leq e^{\pi x^2}$ and $|e^z| = e^{\mathrm{Re}(z)}$, we can bound

$$\begin{cases} |G(z)| \leq e^{\pi r^2 \cos(\theta)^2}, \\ |e^{i\delta z^2}| = e^{-\delta r^2 \sin(2\theta)} \end{cases} \quad |H_\delta(re^{i\theta})| \leq \exp(r^2 \cos\theta(\pi \cos\theta - 2\delta \sin\theta)).$$

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Proof (7/12): Bounding H_δ along ray

Recall

$$|H_\delta(re^{i\theta})| \leq \exp(r^2 \cos\theta(\pi\cos\theta - 2\delta\sin\theta)).$$

For $\theta \leq \pi/2$, $\cos\theta \geq 0$ so negative exponent requires

$$\begin{aligned} \pi\cos\theta - 2\delta\sin\theta < 0 \\ \iff \pi\cos\theta < 2\delta\sin\theta \\ \iff \frac{\pi}{2\delta} < \tan\theta \end{aligned}$$

we can set e.g. $\theta(\delta) = \arctan\left(\frac{\pi}{\delta}\right)$ to guarantee this. As a consequence,

$$\boxed{|H_\delta(re^{i\theta})| \leq 1}$$

Proof (8/12): Apply Phragmén–Lindelöf to H_δ

- Combining $|H_\delta(x)| \leq 1$ and $|H_\delta(re^{i\theta})| \leq 1$, we get that $|H_\delta(z)| \leq 1$ for $z \in \partial S_{\theta(\delta)}$.
- For all $z \in \mathbb{C}$, using $|G(z)| \leq e^{\pi |z|^2}$, we have the growth bound

$$|H_\delta(z)| \leq |e^{i\delta z^2}||G(z)| \leq e^{\delta |z|^2} e^{\pi |z|^2} \leq e^{2\pi |z|^2}$$

  i.e. $\rho = 2 < \frac{\pi}{\theta(\delta) - 0}$.
- Applying Phragmén–Lindelöf yields that $|H_\delta(z)| \leq 1$ for $z \in S_{\theta(\delta)}$ for all small $\delta > 0$.

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Proof (9/12): Extend bound to G on first quadrant

Now fix $z_0 \in S_{\pi/2}$. Since $\theta(\delta) \to \pi/2$ as $\delta \to 0$, there exists $\delta_0 > 0$ such that $z_0 \in S_{\theta(\delta)}$ for all $\delta < \delta_0$.

For any $\varepsilon > 0$ we can find $\delta < \delta_0$ so small that $|e^{-i\delta z_0^2} - 1| < \varepsilon$. Then

$$|H_\delta(z_0)| = |e^{i\delta z_0^2}||G(z_0)| \leq 1 \qquad H_\delta(z) = e^{i\delta z^2}G(z) \quad |H_\delta(z)| \leq 1$$

$$\implies |G(z_0)| \leq |e^{-i\delta z_0^2}| < 1 + \varepsilon \qquad |e^{-i\delta z_0^2} - 1| < \varepsilon$$

Since ε and z_0 were arbitrary, we conclude that $|G(z)| \leq 1$ for all $z \in S_{\pi/2}$.

The same bound holds on $\overline{S_{\pi/2}}$ since $|G(x)| \leq 1$ and $|G(iy)| \leq 1$

Proof (10/12): Modifications for second quadrant

Second quadrant:

- Define $H_\delta(z) = e^{-i\delta z^2} G(z)$ and $\theta(\delta) = \pi - \arctan\left(\frac{\pi}{\delta}\right) \to \pi/2$ as $\delta \to 0$,
- Still $|H_\delta(x)| \leq 1$ and $|H_\delta(z)| \leq e^{2\pi|z|^2}$,
- Consider sector $S_{\theta(\delta)} = \{z \in \mathbb{C} : \theta(\delta) < \arg z < \pi\}$,
- Along ray $z = re^{i\theta}$, $|H_\delta(re^{i\theta})| \leq \exp\left(r^2 \cos\theta(\pi\cos\theta + 2\delta\sin\theta)\right)$,
- By our choice of $\theta(\delta)$, it follows that $|H_\delta(re^{i\theta})| \leq 1$,
- P-L $\implies$ $H_\delta(z) \leq 1$ for $z \in S_{\theta(\delta)}$ for small enough δ,
- Can extend to $|G(z)| \leq 1$ inside entire quadrant by similar argument.

Repeat with similar modifications for the third and fourth quadrants.

$$\boxed{|G(z)| \leq 1 \, \forall z}$$

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Proof (11/12): Applying Liouville's theorem

Having shown that $|G(z)| \leq 1 \, \forall z$ and that G is entire. Liouville's theorem allows us to conclude that G is constant. Now

$$G(z) = C \implies F(z) = Ce^{-\pi z^2} \implies \hat{f}(\omega) = Ce^{-\pi \omega^2} \implies f(x) = Ce^{-\pi x^2}.$$

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Proof (12/12): The $ab > 1$ case

Suppose $|f(x)| \leq e^{-\pi a x^2}$ and $|\hat{f}(\omega)| \leq e^{-\pi b \omega^2}$ for $ab > 1$ and define

$$a_0 = \frac{1}{b} < \frac{ab}{b} = a \implies |f(x)| \leq e^{-\pi a_0 x^2}.$$

Applying the theorem with $a_0 b = 1$ yields

$$f(x) = Ce^{-\pi a_0 x^2}.$$

Combining with the original estimate $|f(x)| \leq e^{-\pi a x^2}$ and $a_0 < a$ yields

$$|f(x)| = Ce^{-\pi a_0 x^2} \leq e^{-\pi a x^2} \implies C \leq e^{-\pi x^2 (a - a_0)} \implies C = 0.$$

□

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Weaker decay / variable exponent

Theorem (Hardy '33)

Let $f \in L^2(\mathbb{R})$ *satisfy* $|f(x)| \leq C_1(1+|x|)^N e^{-\pi a x^2}$ *and* $|\hat{f}(\omega)| \leq C_2(1+|\omega|)^N e^{-\pi b \omega^2}$. *Then*

- $ab > 1 \implies f(x) = 0$,
- $ab = 1 \implies f(x) = P(x)e^{-ax^2}$,

where P *is a polynomial with* $\deg(P) \leq N$.

Theorem (Cowling, Price '82)

Suppose $f \in \mathcal{S}'(\mathbb{R})$ *and*

$$\left\|e^{a\pi x^2} f\right\|_{L^p} + \left\|e^{b\pi\omega^2} \hat{f}\right\|_{L^q} < \infty$$

for $1 \leq p, q \leq \infty$ *where one of* p, q *is* $< \infty$. *Then if* $ab \geq 1$, $f = 0$.

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Higher dimensional version

Obvious generalization to higher dimension also holds.

Theorem (Sitaram, Sundari, Thangavelu '94)

Let $f \in L^2(\mathbb{R}^d)$ *satisfy* $|f(x)| \leq C_1 e^{-\pi a|x|^2}$ *and* $|\hat{f}(\omega)| \leq C_2 e^{-\pi b|\omega|^2}$*. Then*

- $ab > 1 \implies f(x) = 0,$
- $ab = 1 \implies f(x) = Ce^{-a|x|^2}.$

Schrödinger connection

The free Schrödinger equation

$$\partial_t u = i\Delta u,$$
$$u(x,0) = u_0(x)$$

has the general solution

$$u(x,t) = \int_{\mathbb{R}^n} \frac{e^{i|x-y|^2/4t}}{(4\pi i t)^{n/2}} u_0(y)\,dy = \frac{e^{i|x|^2/4t}}{(4\pi i t)^{n/2}} \int_{\mathbb{R}^n} e^{-2ix\cdot y/4t} e^{i|y|^2/4t} u_0(y)\,dy.$$

Let $f(y) = e^{i|y|^2/4t} u_0(y)$, then (up to factors depending on t),

$$|f(x)| = |u_0(x)|, \qquad |\hat{f}(x)| = |u(x,t)|.$$

Here we can apply Hardy!

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Dynamical Hardy

Theorem (Escauriaza–Kenig–Ponce–Vega)

Let u be a solution to the free Schrödinger equation

$$\partial_t u = i\Delta u.$$

Suppose that u is sufficiently smooth and satisfies

$$|u(x,0)| \le C_1\, e^{-\pi a x^2}, \qquad |u(x,T)| \le C_2\, e^{-\pi b x^2}.$$

for some $a, b > 0$. Then

- $ab > \dfrac{1}{(4\pi T)^2} \implies u = 0,$
- $ab = \dfrac{1}{(4\pi T)^2} \implies u(x,t) = Ce^{-\frac{\pi a + i}{4T}|x|^2}.$

Subsequent work looked at other PDE's and added potentials.

Metaplectic version I

The Fourier transform can be realized as a special case of a **metaplectic** operator. They can be realized as compositions of

- Standard Fourier transform $f \mapsto \hat{f}$
- Chirp multiplication $f(t) \mapsto e^{i\pi Qt\cdot t} f(t)$, Q symmetric
- Rescalings $f(t) \mapsto |\det(E)|^{1/2} f(Et)$, E invertible

Includes fractional Fourier transform, chirps and Schrödinger propagators.

$$\text{Metaplectic operator } \hat{S} \leftrightarrow \text{Symplectic matrix } S$$

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{R}^{2d\times 2d}, \qquad A, B, C, D \in \mathbb{R}^{d\times d}.$$

Metaplectic version II

Theorem (Cordero, Giacchi, Malinnikova '24)

Let $f \in L^2(\mathbb{R}^d)$ and $\hat{S}$ be a metaplectic operator such that the associated block matrix B is nonzero, then if M and N are positive-semidefinite matrices with

$$\ker(M) = \ker(B), \qquad R(N) = R(B),$$

f satisfies

$$|f(x)| \leq C_1 e^{-\pi M x \cdot x}, \qquad |\hat{S} f(\xi)| \leq C_2 e^{-\pi N \xi \cdot \xi}$$

and the matrix $MB^T NB$ has an eigenvalue $\lambda > 1$, then $f = 0$.

Enables anisotropic Hardy with regular Fourier transform!

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Thank you!