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Quantum harmonic analysis in time-frequency analysis

What's the deal anyways?

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Setting the stage

QHA from the perspective of time-frequency analysis

- This presentation is supposed to be **easily digestible**
- Basics of time-frequency analysis and how they **relate** to QHA
- Pretty pictures to keep your attention
- Breadth before depth

Heavy theoretical machinery should be heavily motivated, let's try!

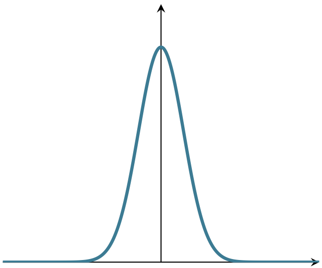
Establishing notation

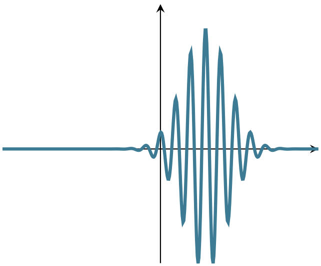
Time-frequency shifts

Our functions $f(t)$, called **signals**, are viewed as time-dependent

We move them in time by **translations** ($T_x : f(t) \mapsto f(t-x)$) and frequency by **modulations** ($M_\omega : f(t) \mapsto e^{2\pi i \omega \cdot t} f(t)$)

$$\pi(x,\omega)f(t) = M_\omega T_x f(t) = e^{2\pi i \omega \cdot t} f(t-x).$$





Establishing notation

Operator translations

For translating operators, we move the function, apply the operator, and then move back the resulting function

$$\alpha_z(S) = \pi(z)S\pi(z)^*.$$

Weyl calculus: There is an isometric isomorphism between functions and operator $A : L^2(\mathbb{R}^{2d}) \to \mathcal{S}^2(L^2(\mathbb{R}^d))$. Operator translations can be viewed as Weyl symbol translation

$$A_{T_z f} = \alpha_z(A_f) \qquad \text{or} \qquad a_{\alpha_z(S)}(x) = a_S(x-z)$$

Defining convolutions

QHA meta statement: Replace

- Functions $\to$ Operators $\quad g \to S$
- Translations $\to$ Operator translations $\quad T_z \to \alpha_z$
- Integrals $\to$ Traces $\quad \int \to \mathrm{tr}$

$$f * g(z) = \int_{\mathbb{R}^{2d}} f(y) T_z \check{g}(y)\, dy$$

$$f \star S = \int_{\mathbb{R}^{2d}} f(z) \alpha_z(S)\, dz$$

(Function-operator convolution)

$$S \star T(z) = \mathrm{tr}(S \alpha_z(\check{T}))$$

(Operator-operator convolution)

Let's investigate the connection to time-frequency analysis!

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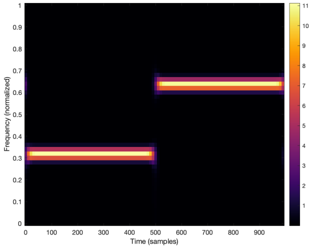
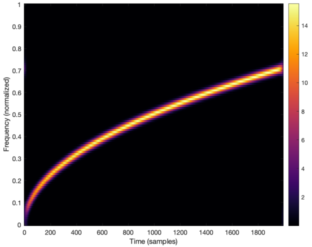
Short-time Fourier transform

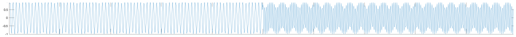
In time-frequency analysis, we use **time-frequency representations** of signals ψ, such as the STFT:

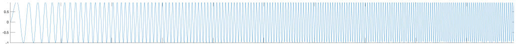
$$V_\varphi \psi(z) = \langle \psi, \pi(z)\varphi \rangle = \int_{\mathbb{R}^d} \psi(t)\overline{\varphi(t-x)}e^{-2\pi i \omega \cdot t}\, dt$$

and its square modulus, the spectrogram:









Localization operators

The STFT $V_\varphi \psi$ contains all the information to reconstruct ψ as

$$\psi = \int_{\mathbb{R}^{2d}} V_\varphi \psi(z) \pi(z) \varphi \, dz.$$

We can add a weight factor (mask, symbol) $m : \mathbb{R}^{2d} \to \mathbb{C}$ to this

$$A_m^\varphi \psi = \int_{\mathbb{R}^{2d}} m(z) V_\varphi \psi(z) \pi(z) \varphi \, dz.$$

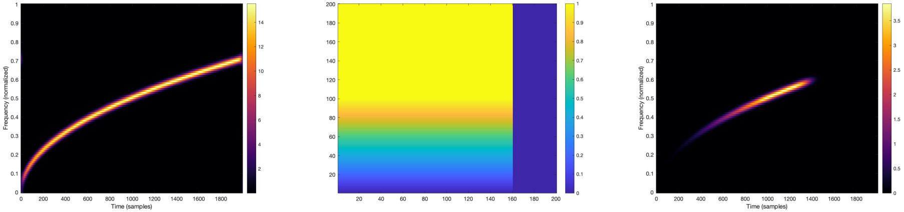


Figure: Original, mask, filtered

Why do we care about localization operators?

- Data augmentation for ML spectrogram classification - SpecAugment by Google (2019), >2800 citations, hot stuff
- Noise reduction - filter noise by masking parts of phase space
- Ideal frequency masks - sound isolation and source separation
- + uses outside time-frequency analysis

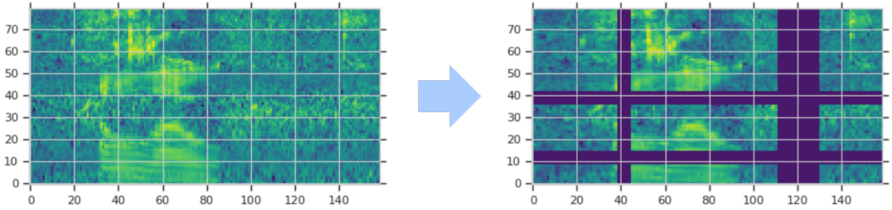


Figure: SpecAugment (Park et al. 2019)

The QHA connection

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Localization operators are rank-one function-operator convolutions!

$$[m \star (\varphi \otimes \varphi)]\psi = \int_{\mathbb{R}^{2d}} m(z)\langle \psi, \pi(z)\varphi\rangle \pi(z)\varphi \, dz = A_m^\varphi \psi$$

What does this give us?

- Conditions for when $\{A_m^\varphi : m \in L^p\}$ is dense in $\mathcal{S}^p$
- A procedure to approximate m from A_m^φ
- A simple perspective on standard properties ($\|A_m^\varphi\|_{\mathcal{S}^p} \leq \|m\|_{L^p}\|\varphi\|_{L^2}^2$) - which is easy to extend to wavelets
- A way to lift mask convergence to operator convergence

Properties of localization operators

- Integral of convolution is product of integrals

$$\operatorname{tr}(f \star S) = \int_{\mathbb{R}^{2d}} f(z)\,dz \cdot \operatorname{tr}(S) \implies \operatorname{tr}(A_m^\varphi) = \int_{\mathbb{R}^{2d}} m(z)\,dz$$

- Reconstruction with constant mask

$$1 \star S = \operatorname{tr}(S) \cdot I_{L^2} \implies A_1^\varphi = I_{L^2}$$

Weyl symbols respect convolutions:

$$a_{f\star S} = m * a_S \implies a_{A_m^\varphi} = m * W(\varphi)$$

where $W(\varphi) = a_{\varphi\otimes\varphi}$

Through the SVD $S = \sum_n \lambda_n(\phi_n \otimes \phi_n)$, general function-operator convolutions can be realized as **mixed-state** localization operators

$$m \star S = \sum_n s_n m \star (\phi_n \otimes \phi_n).$$

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A topical deep dive

From Gabor multipliers to localization operators

Let $\Lambda_{\alpha,\beta} = \alpha\mathbb{Z} \times \beta\mathbb{Z}$ (rectangular lattice) and $\mu^m_{\alpha,\beta} = \alpha\beta \sum_{\lambda \in \Lambda_{\alpha,\beta}} m(\lambda)\delta_\lambda$ (discretization), then

$$\left[\mu^m_{\alpha,\beta} \star (\varphi \otimes \varphi)\right]\psi = \alpha^d \beta^d \sum_{\lambda \in \Lambda_{\alpha,\beta}} m(\lambda) V_\varphi \psi(\lambda) \pi(\lambda)\varphi.$$

Letting $\alpha, \beta \to 0$,

$$\underbrace{\mu^m_{\alpha,\beta} \star (\varphi \otimes \varphi)}_{\text{Gabor multiplier}} \to \underbrace{m \star (\varphi \otimes \varphi)}_{\text{Localization operator}} \qquad \text{in } \mathcal{S}^1.$$

Proof is made much easier by considering general function-operator convolutions and then specializing to the rank-one case.

Intermission

Without having to resort to internal soul-searching, we can get operator-operator convolutions from function operator convolutions:

Adjoints:
Let $\mathcal{A}_S : L^p \to \mathcal{S}^p$ with $f \mapsto f \star S$, then

$\mathcal{A}_S^* : \mathcal{S}^p \to L^p$ with $T \mapsto T \star S$

Weyl calculus respect convolutions:

$$S \star T(z) = a_S * a_T(z)$$

(This of course also works the other way around)

Operator-operator convolutions

Spectrograms are rank-one operator-operator convolutions!

$$(\psi \otimes \psi) \star (\check{\varphi} \otimes (\check{\varphi})(z) = \operatorname{tr}\left[(\psi \otimes \psi)\pi(z)(\varphi \otimes \varphi)\pi(z)\right]$$
$$= |\langle \psi, \pi(z)\varphi \rangle|^2.$$

But it turns out the full-rank case is also of interest!

$$(\psi \otimes \psi) \star \check{S}(z) = Q_S(\psi)(z)$$

This class is **Cohen's class of quadratic time-frequency distributions**

Cohen's class was originally defined as

$$Q_\Phi(\psi)(z) = W(\psi) * \Phi(z) = A_{W(\psi)} \star A_\Phi(z)$$

so $S = A_\Phi$ and $A_{W(\psi)} = \psi \otimes \psi$ via the Weyl calculus.

Obvious question: Why do we care about Cohen's class?

Why do we care about Cohen's class?

Essentially; why don't we just always use the spectrogram?

- Other TF distributions can be more sparse, meaning separating signals is easier
- We can give ML models access to different TF distributions to give them more (effective) data and improve performance
- Distributions are tailored to applications

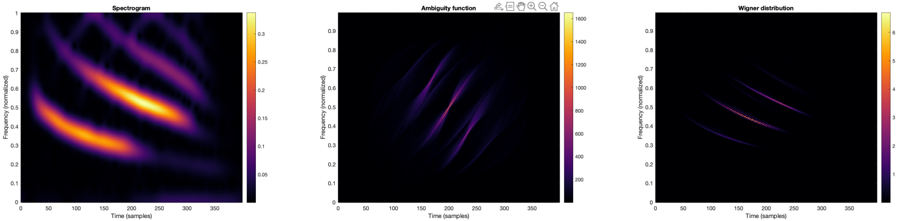

Figure: Same signal; Spectrogram, Ambiguity function and Wigner distribution

Basic properties of Cohen's class

- Energy preserving

$$\int_{\mathbb{R}^{2d}} S \star T(z)\, dz = \operatorname{tr}(S)\operatorname{tr}(T) \implies \int_{\mathbb{R}^{2d}} Q_S(\psi)(z)\, dz = 1$$

- Uncertainty principle

$$\int_{\Omega} Q_S(\psi)(z)\, dz > 1 - \varepsilon \implies |\Omega| > 1 - \varepsilon$$

- By performing a SVD on S, we see that

$$Q_S(\psi) = (\psi \otimes \psi) \star \sum_n s_n(\phi_n \otimes \phi_n) = \sum_n s_n |V_{\phi_n}\psi|^2$$

- Characterized as continuous mappings $L^2(\mathbb{R}^d) \to C_b(\mathbb{R}^{2d})$ which respect translations

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A Fourier transform

We wish to do some harmonic analysis with these distributions - hence we need a Fourier transform

$$\mathcal{F}_W(S)(z) = e^{-i\pi x\cdot\omega} \operatorname{tr}\big(\pi(-z)S\big), \qquad \mathcal{F}_W : \mathcal{S}^1 \to C_0(\mathbb{R}^{2d})$$

Together with the *symplectic* Fourier transform $\mathcal{F}_\sigma$, we have a convolution theorem

$$\mathcal{F}_W(f \star S) = \mathcal{F}_\sigma(f) \cdot \mathcal{F}_W(S),$$
$$\mathcal{F}_\sigma(T \star S)(z) = \mathcal{F}_W(T)(z) \cdot \mathcal{F}_W(S)(z).$$

Weyl symbols can be realized as:

$$a_S = \mathcal{F}_\sigma(\mathcal{F}_W(S)).$$

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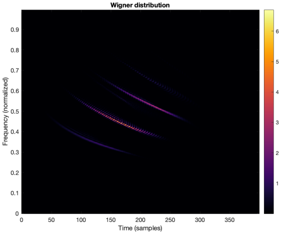
Cohen's class phase retrieval

Questions: Does $Q_S(\psi)$ uniquely determine ψ?



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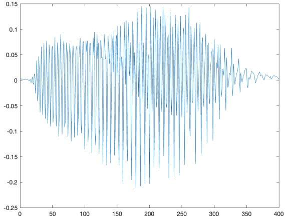

Answer: Sometimes,

$$\mathcal{F}_\sigma(Q_S(\psi)) = \mathcal{F}_\sigma\left[(\psi \otimes \psi) \star \check{S}\right] = \mathcal{F}_W(\psi \otimes \psi) \cdot \mathcal{F}_W(\check{S})$$

so if $\mathcal{F}_W(\check{S})$ is non-zero we can recover $\mathcal{F}_W(\psi \otimes \psi) \rightarrow (\psi \otimes \psi) \rightarrow \psi$.

(Phase retrieval is basically deconvolution, explains instability)

Cohen induces mixed-state localization operators

Alternatively; op-op convolutions induce func-op convolutions

Solving the optimization problem

$$\max_{\|\psi\|=1} \int_{\mathbb{R}^{2d}} m(z)|V_\varphi \psi(z)|^2 \, dz$$

gives rise to localization operators A_m^φ via orthogonal maximizers.

The corresponding Cohen's class problem

$$\max_{\|\psi\|=1} \int_{\mathbb{R}^{2d}} m(z) Q_S(\psi)(z) \, dz$$

gives $m \star S$.

By SVD, $m \star S$ for finite rank window operator S is a so called multi-window localization operator

(My) key takeaways

- QHA provides another lens through which time-frequency analysis can be investigated
- With this additional lens, the number of facts which are "clear" is strictly increased
- The fact that so much of the structure and intuition from harmonic survives (and is useful) in this setting is remarkable!

... and QHA plays well with **Weyl calculus**!

That’s it, now questions then lunch!