Measure-operator convolutions and applications to mixed-state Gabor multipliers

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joint work with Hans Feichtinger and Franz Luef, published in *Sampling Theory, Signal Processing, and Data Analysis*

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The short version

We have seen plenty about function-operator convolutions throughout the workshop

$$f \star S = \int_{\mathbb{R}^{2d}} f(z)\pi(z)S\pi(z)^* \, dz.$$

Clearly to generalize this to measure-operators convolutions, we can go with

$$\mu \star S = \int_{\mathbb{R}^{2d}} \pi(z)S\pi(z)^* \, d\mu(z),$$

all in a day's work!

Thank you! Questions?

Nah!

What's wrong?

$$\mu \star S = \int_{\mathbb{R}^{2d}} \pi(z) S \pi(z)^* \, d\mu(z)$$

It's fine to define measure-operator convolutions this way:

- Define as Bochner integral - just as for function-operator convolutions
- Define via Weyl symbol - allows large class of tempered distributions

There is further happiness to gain:

- A "first principles" approach is nice - rederive QHA
- Get free properties from associated framework
- End goal is establishing new results outside of QHA (spoiler)

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Enabler/starting point

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Abstract nonsense:

H. G. Feichtinger
A Novel Mathematical Approach to the Theory of Translation Invariant Linear Systems
Recent Applications of Harmonic Analysis to Function Spaces, Differential Equations, and Data Science: Novel Methods in Harmonic Analysis, Volume 2, Springer International Publishing, Cham, 2017, pp. 483–516.

H. G. Feichtinger
Homogeneous Banach spaces as Banach convolution modules over $M(G)$
Mathematics, 10(3), 364, 2022, MDPI AG.

Classical analogy I

How could your grandparents have defined convolution? Obviously **homogenous Banach spaces**, via translations!

$$\rho : \mathbb{R}^d \ni x \mapsto T_x \in B(L^1)$$

The representation ρ is

- Linear $(\rho(x)(\alpha f + \beta g) = \alpha\rho(x)f + \beta\rho(x)g)$
- Preserves identity $(\rho(0)f = f)$
- Group homomorphism $(\rho(x + y) = \rho(x)\rho(y))$
- Isometric $(\|\rho(x)f\|_{L^1} = \|f\|_{L^1})$
- Continuous $(\|\rho(x)f - f\|_{L^1} \to 0$ as $x \to 0)$

and we say that (L^1, ρ) is a homogenous Banach space.

Classical analogy II

We can define the action $*_\rho : \mathbb{R}^d \times L^1 \to L^1$ as

$$*_\rho : (x, f) \mapsto T_x f$$

or, on point measures, $*_\rho : (\delta_x, f) \mapsto T_x f$. By **some functional analysis**, this action can be extended to $M(\mathbb{R}^d) \times L^1$ to define convolutions between bounded measures and integrable functions.

Upside? Limited

Operator version I

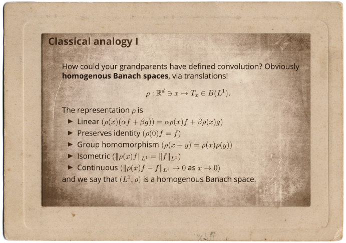

Figure: Never forget your roots

Let's translate this to operators!

Translations $\to$ operator translations, functions $\to$ operators:

$$\rho : \mathbb{R}^{2d} \ni z \mapsto \alpha_z \in B(\mathcal{S}^1),$$
$$\alpha_z(S) = \pi(z)S\pi(z)^*.$$

- Linear $(\rho(z)(\alpha S_1 + \beta S_2)) = \alpha\rho(z)S_1 + \beta\rho(z)S_2)$
- Preserves identity $(\rho(0)S = S)$
- Group homomorphism $(\rho(z_1 z_2) = \rho(z_1)\rho(z_2))$
- Isometric $(\|\rho(z)S\|_{\mathcal{S}^1} = \|S\|_{\mathcal{S}^1})$
- Continuous $(\|\rho(z)S - S\|_{\mathcal{S}^1} \to 0$ as $z \to 0)$

we say that $(\mathcal{S}^1, \rho)$ is an **abstract homogenous Banach space**.

Operator version II

Applying the same functional-analytical machinery allows us to (uniquely) extend the mapping

$$*_\rho : \mathbb{R}^{2d} \times \mathcal{S}^1 \to \mathcal{S}^1, \qquad z *_\rho S = \pi(z) S \pi(z)^*$$

to one on $M(\mathbb{R}^{2d}) \times \mathcal{S}^1$ (which is bounded, bilinear, w^*-continuous and has dense span) using BUPU's:

$$\mu *_\rho S = \lim_{|\Psi| \to 0} \sum_{i \in I_\Psi} \mu(\psi_i) \delta_{z_i} *_\rho S.$$

We call this **measure-operator convolutions** and write $\star$ for $*_\rho$.

The BUPU machinery allows us to ultimately derive the formula:

$$\langle (\mu \star S)\psi, \phi \rangle = \int_{\mathbb{R}^{2d}} \langle \pi(z) S \pi(z)^* \psi, \phi \rangle \, d\mu(z).$$

What now?

TODO:
We should prove that all the standard function-operator properties hold true

$\leftarrow$ medium fun

- $\|\mu \star S\|_{\mathcal{S}^p} \leq \|\mu\|_M \|S\|_{\mathcal{S}^p}$
- $\mu \star S \geq 0$ if $\mu \geq 0$ and $S \geq 0$
- $\operatorname{tr}(\mu \star S) = \mu(\mathbb{R}^{2d}) \operatorname{tr}(S)$ when $S \in \mathcal{S}^1$
- $(\mu \star S)\check{} = \check{\mu} \star \check{S}$
- $\mathcal{F}_W(\mu \star S) = \mathcal{F}_\sigma(\mu) \cdot \mathcal{F}_W(S)$
- $\vdots$

Essentially all we can dream of is true - this makes subsequent work easier

Not-so-basic property

The main payoff of using this framework is essentially the following theorem:

Theorem
Let $(\mu_\alpha)_\alpha$ *be a bounded and tight net which converges weak-* to* μ_0 *and* $S \in \mathcal{S}^1$*, then*

$$\lim_{\alpha\to\infty} \|\mu_\alpha \star S - \mu_0 \star S\|_{\mathcal{S}^1} = 0.$$

(Recall this means that $\mu_\alpha(f) \to \mu_0(f)$ for all $f \in M(\mathbb{R}^{2d})^* = C_b(\mathbb{R}^{2d})$)

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Part II: Contributing to society

The lattice setting

We are interested in cases where

$$\mu = \sum_{\lambda \in \Lambda} c(\lambda)\delta_\lambda \implies \mu \star S = \sum_{\lambda \in \Lambda} c(\lambda)\alpha_\lambda(S)$$

for some lattice $\Lambda \subset \mathbb{R}^{2d}$.

This is (often) the setting of discrete time-frequency analysis as it is straightforward to implement numerically ($\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$).

These operators were previously investigated by Skrettingland with the notation $c \star_\Lambda S$:

Eirik Skrettingland
Quantum Harmonic Analysis on Lattices and Gabor Multipliers
Journal of Fourier Analysis and Applications, 26(3), 2020, Springer.

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Mixed-state Gabor frames

Recall that (g, Λ) generates a **Gabor frame** when

$$A\|f\|^2 \leq \sum_{\lambda \in \Lambda} |V_g f(\lambda)|^2 \leq B\|f\|^2 \qquad \forall f \in L^2(\mathbb{R}^d).$$

We say that (S, Λ) generates a ***mixed-state* Gabor frame** when

$$A\|f\|^2 \leq \sum_{\lambda \in \Lambda} |Q_S(f)(\lambda)|^2 \leq B\|f\|^2 \qquad \forall f \in L^2(\mathbb{R}^d).$$

If $A = B$, we have a nice reconstruction of the identity:

$$\sum_{\lambda \in \Lambda} \pi(\lambda) S \pi(\lambda)^* f = Af \qquad \forall f \in L^2(\mathbb{R}^d).$$

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(Mixed-state) Gabor multipliers

Tight Gabor frame $\implies$ reconstruction formula

$$f = \sum_{\lambda \in \Lambda} V_g f(\lambda) \pi(\lambda) g$$

which gives rise to **Gabor multipliers**

$$G^g_{m,\Lambda} f = \sum_{\lambda \in \Lambda} m(\lambda) V_g f(\lambda) \pi(\lambda)$$

with mask m.

Tight mixed-state Gabor frame $\implies$ reconstruction formula

$$f = \sum_{\lambda \in \Lambda} \pi(\lambda) S \pi(\lambda)^* f$$

which gives rise to ***mixed-state* Gabor multipliers**

$$G^S_{m,\Lambda} f = \sum_{\lambda \in \Lambda} m(\lambda) \pi(\lambda) S \pi(\lambda)^* f$$

with mask m.

It turns out (perhaps expectedly) that these operators behave similarly to the usual Gabor multipliers.

0-1 Gabor multiplier eigenvalue law

- The eigenvalues of localization operators famously follow a 0-1 law where if $m = \chi_\Omega$, the first $\lceil |\Omega| \rceil$ eigenvalues of A^g_Ω are close to 1 and the remaining eigenvalues are close to 0.
- This is easiest to prove using QHA.
- With measure-operator convolutions, we can follow the same path for mixed-state Gabor multipliers.

Theorem

Let (S, Λ) generate a tight mixed-state Gabor frame, let $\Omega \subset \mathbb{R}^{2d}$ be compact and fix $\delta \in (0,1)$. If $\{\lambda_k^{R\Omega}\}_k$ are the eigenvalues of $G^S_{R\Omega,\Lambda}$, then

$$\frac{\#\{k : \lambda_k^{R\Omega} > 1 - \delta\}}{|R\Omega \cap \Lambda|} \to 1 \quad \text{as } R \to \infty.$$

Painless QHA on lattices™

Approximating localization operators

Ideally, we want our discrete constructions to approximate our continuous constructions in some limit.

Define:

$$\mu^m_{\alpha,\beta} = \alpha^d \beta^d \sum_{\lambda \in \Lambda_{\alpha,\beta}} m(\lambda)\delta_\lambda$$

where $\Lambda_{\alpha,\beta} = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$.

Theorem

Let $(\mu_\alpha)_\alpha$ be a bounded and tight net which converges weak- to μ_0 and $S \in \mathcal{S}^1$, then*

$$\lim_{\alpha\to\infty} \|\mu_\alpha \star S - \mu_0 \star S\|_{\mathcal{S}^1} = 0.$$

Theorem

Let $m \in W(L^\infty, \ell^1)(\mathbb{R}^{2d})$ be Riemann-integrable and $S \in \mathcal{S}^1$. Then we have the convergence

$$\lim_{\alpha,\beta\to 0} \left\|\mu^m_{\alpha,\beta} \star S - m \star S\right\|_{\mathcal{S}^1} = 0.$$

In particular, $\|G^g_{m,\alpha,\beta} - A^g_m\|_{\mathcal{S}^1} \to 0$ as $\alpha, \beta \to 0$.

Why does this work?

Verifying the convergence

$$\mu^m_{\alpha,\beta}(f) \to \int_{\mathbb{R}^{2d}} m(z) f(z)\, dz$$

boils down to realizing the left-hand side

$$\mu^m_{\alpha,\beta}(f) = \sum_{\lambda \in \Lambda} m(\lambda) f(\lambda) \alpha^d \beta^d$$

as a Riemann sum.

We also need to verify that $(\mu^m_{\alpha,\beta})_{\alpha,\beta}$ is tight and uniformly bounded (harder than it looks).

Theorem
Let $(\mu_\alpha)_\alpha$ be a bounded and tight net which converges weak- to μ_0 and $S \in \mathcal{S}^1$, then*

$$\lim_{\alpha \to \infty} \|\mu_\alpha \star S - \mu_0 \star S\|_{\mathcal{S}^1} = 0.$$

Theorem
Let $m \in W(L^\infty, \ell^1)(\mathbb{R}^{2d})$ be Riemann-integrable and $S \in \mathcal{S}^1$. Then we have the convergence

$$\lim_{\alpha,\beta \to 0} \left\| \mu^m_{\alpha,\beta} \star S - m \star S \right\|_{\mathcal{S}^1} = 0.$$

Parameter continuity

Theorem
Suppose that as $n \to \infty$,

$$\begin{cases} \alpha_n \to \alpha & \text{in } \mathbb{R}^2 \\ \beta_n \to \beta & \text{in } \mathbb{R}^2 \\ m_n \to m & \text{in } W(C_0, \ell^1)(\mathbb{R}^d) \\ S_n \to S & \text{in } \mathcal{S}^1 \end{cases} \implies G^{S_n}_{m_n,\alpha_n,\beta_n} \to G^{S}_{m,\alpha,\beta} \quad \text{in } \mathcal{S}^1.$$

"Gabor multipliers are $\mathcal{S}^1$-continuous with respect to their parameters".
Earlier results have been limited to $\mathcal{S}^2$ convergence or $g \in \mathcal{S}(\mathbb{R}^d)$.

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(Actual) Thank you!