Quantum harmonic analysis on locally compact groups

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Introduction

A square integrable representation of a locally compact group induces a natural notion of function-operator and operator-operator convolutions. These generalize the convolutions defined in Werner's theory of quantum harmonic analysis on phase space. As a result, we can deduce properties of localization operators, Cohen's class distributions and related objects using a common framework since these can be realized as convolutions. These convolutions have recently been studied in the case where the representation corresponds to time-frequency shifts and time-scale shifts but not in the more general setting of locally compact groups.

In the left column we collect some general properties while in the right column we state the main results.

Basics of quantum harmonic analysis

Quantum harmonic analysis, introduced in Werner 1984, is focused on extending results from harmonic analysis to the operator setting with the Schatten p-class of operators replacing the L^p space of functions. This convention goes hand in hand with letting traces replace integrals for measuring the "size" of an operator. To translate operators and generalize the classical translation $T_x f(t) = f(t-x)$, the *operator translation*

$$\alpha_z(S) = \pi(z)^* S \pi(z)$$

is defined, where π is the standard time-frequency representation. This in turns allows us to define convolutions between functions and operators and operators and operators as

$$f \star S = \int_{\mathbb{R}^{2d}} f(z)\alpha_z(S)\,dz, \qquad T \star S(z) = \mathrm{tr}(T\alpha_z(S)).$$

It turns out that this is a very fruitful approach both for simplifying classical theorems and showing new connections.

Key definitions

In the locally compact setting, we let G denote an arbitrary locally compact group with square integrable representation $\sigma : G \to \mathcal{U}(\mathcal{H})$ where $\mathcal{H}$ is a Hilbert space and μ_r is the right Haar measure. Let $f \in L^1_r(G)$, $S \in \mathcal{S}^1$ and $T \in B(\mathcal{H})$, we then define

- **Function-operator convolution:**

$$f \star_G S = \int_G f(x)\sigma(x)^* S \sigma(x)\,d\mu_r(x)$$

- **Operator-operator convolution:**

$$T \star_G S(x) = \mathrm{tr}(T\sigma(x)^* S\sigma(x))$$

Connection to time-frequency analysis

Much, but not all, of the utility of quantum harmonic analysis comes from its connection to time-frequency analysis. This comes mainly in the form of the following two objects.

- **Localization operators:** When $S = \varphi_1 \otimes \varphi_2$, the function-operator convolution $f \star_G S$ is precisely the localization operator $\mathcal{A}_f^{\varphi_1,\varphi_2}$. The generalization to arbitrary trace-class S is known as *mixed-state* localization operators or *multiwindow* STFT multipliers.
- **Cohen's class distributions:** When $T = \psi \otimes \phi$, the operator-operator convolution $T \star_G S$ is precisely the Cohen's class distribution

$$Q_S(\psi,\phi)(z) = (\psi \otimes \phi) \star_G S(x).$$

In particular, when S is a rank-one operator we recover the spectrogram.

Admissibility of operators

Integrability of operator-operator convolutions is key to all results which generalize results built on Moyal's identity. In the locally compact setting, this means generalizing the Duflo-Moore orthogonality relations. If $\mathcal{D}^{-1}$ is the Duflo-Moore operator, then

$$\int_G T \star_G S(x)\,d\mu_r(x) = \mathrm{tr}(T)\,\mathrm{tr}(\mathcal{D}^{-1}S\mathcal{D}^{-1}).$$

Operators $S \in \mathcal{S}^1$ which satisfy $\mathcal{D}^{-1}S\mathcal{D}^{-1} \in \mathcal{S}^1$ are said to be **admissible**.

A general operator $S = \sum_n s_n \xi_n \otimes \xi_n$ is admissible if and only if each ξ_n is admissible in the sense of Duflo-Moore and

$$\sum_n s_n \|\mathcal{D}^{-1}\xi_n\|^2 < \infty.$$

Motivating examples of locally compact groups

Quantum harmonic analysis is usually carried out on the Weyl-Heisenberg groups and was recently extended to the affine group in Berge et al. 2022. We list some motivating examples of why we want to extend quantum harmonic analysis to arbitrary locally compact groups with square integrable representations.

- **Affine group** $\mathrm{Aff} = (\mathbb{R} \times \mathbb{R}^+, \cdot_{\mathrm{Aff}})$, with the square integrable representation

$$\pi(x,a)f(t) = e^{2\pi i x t}\psi(at)$$

and Haar measures $d\mu_\ell(x,a) = \frac{dx\,da}{a^2}$, $d\mu_r(x,a) = \frac{dx\,da}{a}$.

- **Shearlet group** $\mathbb{S} = (\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2, \cdot_{\mathbb{S}})$, with the square integrable representation

$$\pi(a,s,x)\psi(t) = T_x D_{S_s A_a}\psi(t) = a^{-3/4}\psi(A_a^{-1}S_s^{-1}(t-x))$$

and Haar measures $d\mu_\ell(a,s,x) = \frac{da\,ds\,dx}{a^3}$, $d\mu_r(a,s,x) = \frac{da\,ds\,dx}{a}$.

- **Similitude group** $SIM(2) = (\mathbb{R}^+ \times \mathbb{R}^2 \times SO(2), \cdot_{SIM(2)})$, with the square integrable representation

$$\pi(a,x,\theta)\psi(t) = a^{-1}\psi\Big(\tau_{-\theta}\Big(\frac{t-x}{a}\Big)\Big)$$

and Haar measures $d\mu_\ell(a,x,\theta) = \frac{da\,dx\,d\theta}{a^3}$, $d\mu_r(a,x,\theta) = \frac{da\,dx\,d\theta}{a}$.

- **Higher dimensional analogues**. Both the shearlet and similitude groups have higher dimensional analogues which give rise to square integrable representations.

Interpolated convolution mapping properties

Both function-operator and operator-operator convolutions satisfy generalizations of Young's inequality. In the following, $\frac{1}{p}+\frac{1}{q}=1$ and the membership of f, S, T is differs betwen inequalities.

$$\|f \star_G S\|_{\mathcal{S}^p} \le \|f\|_{L^1_r(G)}\|S\|_{\mathcal{S}^p},$$
$$\|f \star_G S\|_{\mathcal{S}^p} \le \|f\|_{L^p_r(G)}\|S\|_{\mathcal{S}^1}^{1/p}\|\mathcal{D}^{-1}S\mathcal{D}^{-1}\|_{\mathcal{S}^1}^{1/q},$$
$$\|T \star_G S\|_{L^\infty(G)} \le \|S\|_{\mathcal{S}^p}\|T\|_{\mathcal{S}^q},$$
$$\|T \star_G S\|_{L^p_r(G)} \le \|T\|_{\mathcal{S}^p}\|S\|_{\mathcal{S}^1}^{1/q}\|\mathcal{D}^{-1}S\mathcal{D}^{-1}\|_{\mathcal{S}^1}^{1/p}.$$

The second and fourth inequality requires that S is **admissible**. Moreover, if S is **admissible**, then the (bounded) mappings

$$\mathcal{A}_S : L^p_r(G) \to \mathcal{S}^p, \qquad f \mapsto f \star_G S,$$
$$\mathcal{B}_S : \mathcal{S}^p \to L^p_r(G), \qquad T \mapsto T \star_G S$$

are adjoints of each other.

Eigenvalues of mixed-state affine localization operators

Mixed-state localization operators are a higher dimensional analogue of localization operators, defined for $S = \sum_n s_n \varphi_n \otimes \varphi_n$ and a subset $\Omega \subset G$ as

$$\chi_\Omega \star S = \sum_n s_n \mathcal{A}_\Omega^{\varphi_n}$$

where $\mathcal{A}_\Omega^{\varphi_n}$ is a classical localization operators. We are able to prove the following result on the eigenvalues of such operators.

Theorem:
*Let S be an **admissible** trace-class operator with $\mathrm{tr}(\mathcal{D}^{-1}S\mathcal{D}^{-1}) = 1$ on $L^2(\mathbb{R}^+)$, let $\Omega \subset \mathrm{Aff}$ be a compact domain and fix $\delta \in (0,1)$. If $\{\lambda_k^{R\Omega}\}_k$ are the non-zero eigenvalues of $\chi_{R\Omega} \star_{\mathrm{Aff}} S$, then*

$$\frac{\#\{k : \lambda_k^{R\Omega} > 1-\delta\}}{\mathrm{tr}(S)\mu_r(R\Omega)} \to 1 \quad \text{as } R \to \infty.$$

Berezin-Lieb inequalities

Fix a positive $T \in \mathcal{S}^1$ and let $S \in \mathcal{S}^1$ be **admissible**. If Φ is a non-negative, convex and continuous function on a domain containing the spectrum of $\mathrm{tr}(S)T$ and the range of $T \star_G S$, then

$$\int_G \Phi \circ (T \star_G S)(x)\,d\mu_r(x) \le \mathrm{tr}\,(\Phi(\mathrm{tr}(S)T)\frac{\mathrm{tr}(\mathcal{D}^{-1}S\mathcal{D}^{-1})}{\mathrm{tr}(S)}$$

Similarly, if $S \in \mathcal{S}^1$ is positive and **admissible**, $f \in L^\infty(G)$ is non-negative and Φ is a non-negative, convex, and continuous function on a domain containing the spectrum of $f \star_G S$ and the range of $\mathrm{tr}(\mathcal{D}^{-1}S\mathcal{D}^{-1})f$, then

$$\mathrm{tr}(\Phi(f \star_G S)) \le \frac{\mathrm{tr}(S)}{\mathrm{tr}(\mathcal{D}^{-1}S\mathcal{D}^{-1})}\int_G \Phi\big(\mathrm{tr}(\mathcal{D}^{-1}S\mathcal{D}^{-1})f(x)\big)\,d\mu_r(x).$$

Wiener's Tauberian theorem

We first make the following two definitions:
A function $g \in L^p_r(G)$ is said to be p-regular if

$$\overline{\mathrm{span}\,\{g(\cdot\, x^{-1})\}_{x\in G}} = L^p_r(G).$$

Similarly, an operator $S \in \mathcal{S}^p$ is said to be p-regular if

$$\overline{\mathrm{span}\,\{\sigma(x)^* S\sigma(x)\}_{x\in G}} = \mathcal{S}^p.$$

Now assume that there exists an admissible operator $R \in \mathcal{S}^1$ such that $R \star_G R$ is regular, let $S \in \mathcal{S}^p$ be **admissible**, $1 \le p \le \infty$ and let q be the conjugate exponent of p. Then the following are equivalent:

1. S is p-regular,
2. If $f \in L^q_r(G)$ and $f \star_G S = 0$, then $f = 0$,
3. $\mathcal{S}^p \star_G S$ is dense in $L^p_r(G)$,
4. If $T \in \mathcal{S}^q$ and $T \star_G S = 0$, then $T = 0$,
5. $L^p_r(G) \star_G S$ is dense in $\mathcal{S}^p$,
6. $S \star_G S$ is p-regular,
7. For any regular $T \in \mathcal{S}^1, T \star_G S$ is p-regular.

References

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